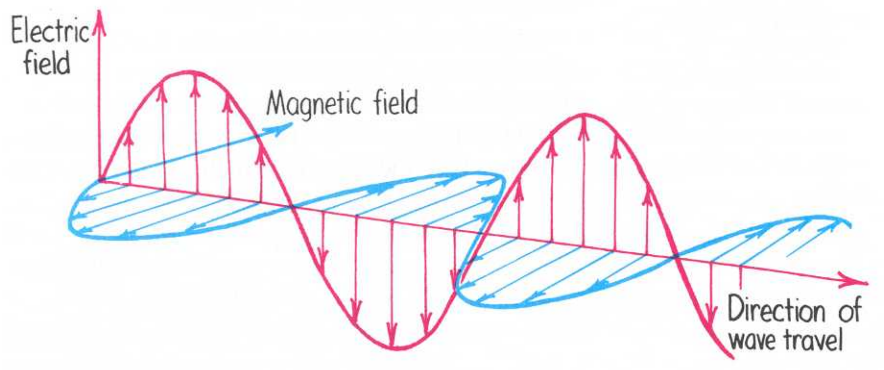
PHY2009S
“Fields and fluids”

# Part C: A brief introduction to Maxwell’s equations

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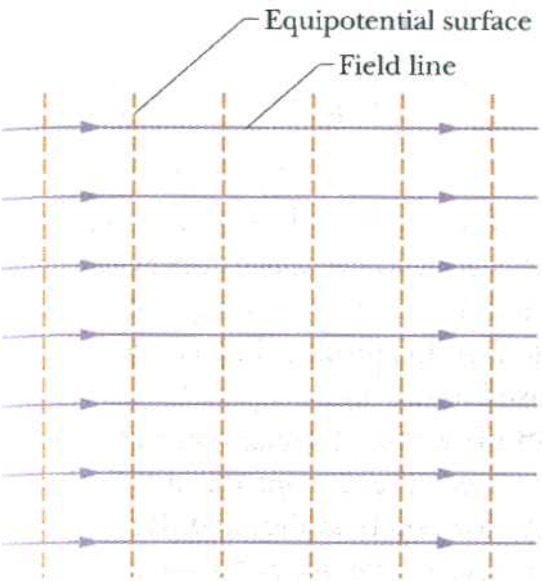
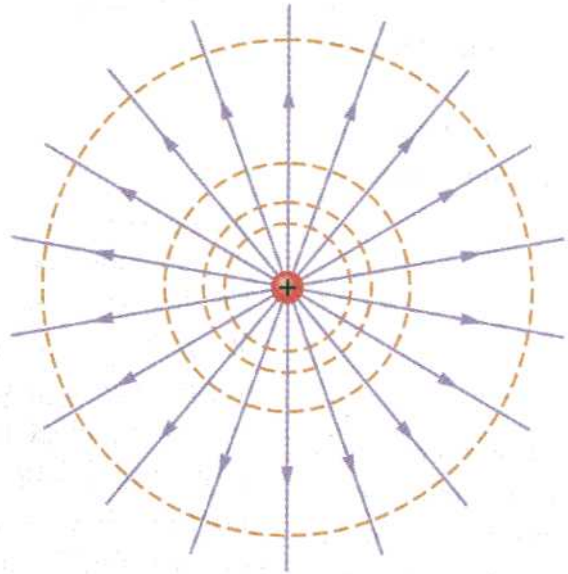
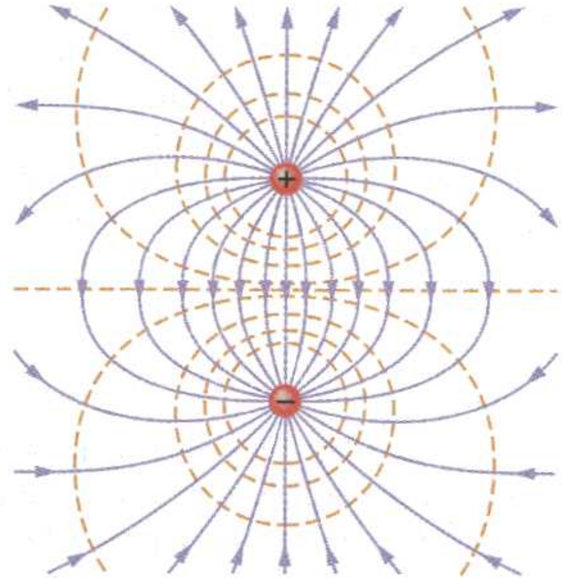
Just about the whole of electromagnetism is contained in the Maxwell equations:

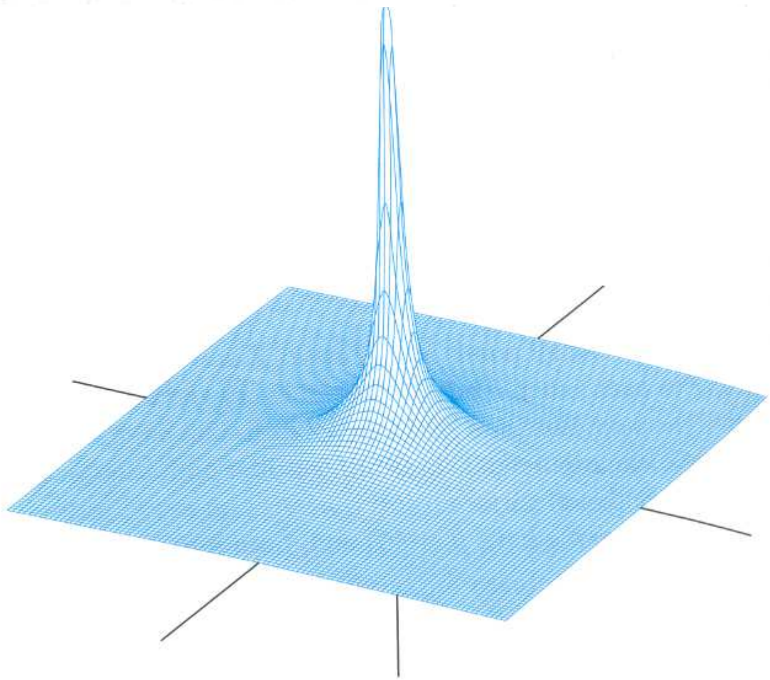
$$\vec{\nabla}\bullet\vec{\mathbf{E}} = \frac{\rho}{\varepsilon_o} \qquad \vec{\nabla}\bullet\vec{\mathbf{B}} = 0$$

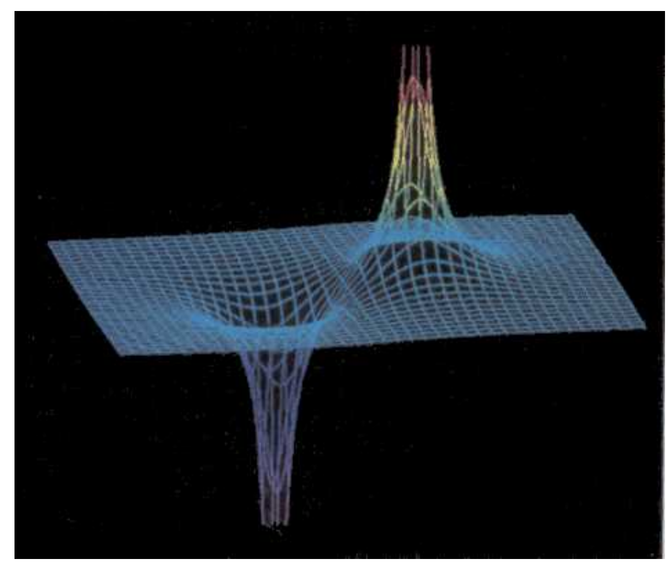
$$\vec{\nabla}\times\vec{\mathbf{E}} = -\frac{\partial\vec{\mathbf{B}}}{\partial t} \qquad \vec{\nabla}\times\vec{\mathbf{B}} = \mu_o\vec{\mathbf{j}} + \mu_o\varepsilon_o\frac{\partial\vec{\mathbf{E}}}{\partial t}$$


Equipotential surface
Field line







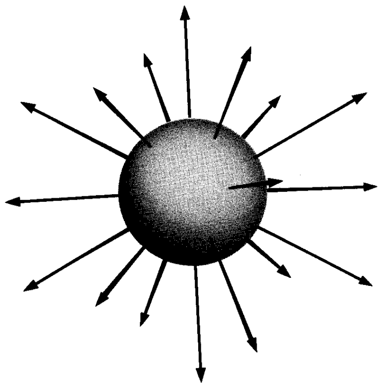


# Gauss' law of electrostatics

For several discrete charges $\oint_S \vec{\mathbf{E}} \bullet d\vec{\mathbf{a}} = \frac{\sum q_i}{\varepsilon_o}$

The electric flux through a closed surface *S* is proportional to the total charge enclosed within *S*.

For a continuous charge distribution $\rho(\vec{\mathbf{r}})$



$$\oint_S \vec{\mathbf{E}} \bullet d\vec{\mathbf{a}} = \frac{1}{\varepsilon_o} \int_V \rho(\vec{\mathbf{r}}) dV$$

More specifically, the net outward flux of the electric field across any closed surface *S* is equal to the total electric charge enclosed within the region *V* bounded by *S*, divided by the permittivity of free space $\varepsilon_o$.

# Gauss' law of electrostatics ... 2

By the divergence theorem: $\int_S \vec{\mathbf{E}} \bullet d\vec{\mathbf{a}} = \int_V \operatorname{div} \vec{\mathbf{E}}\, dV$

$$\therefore \int_V \operatorname{div} \vec{\mathbf{E}}\, dV = \frac{1}{\varepsilon_o} \int_V \rho(\vec{\mathbf{r}})\, dV$$

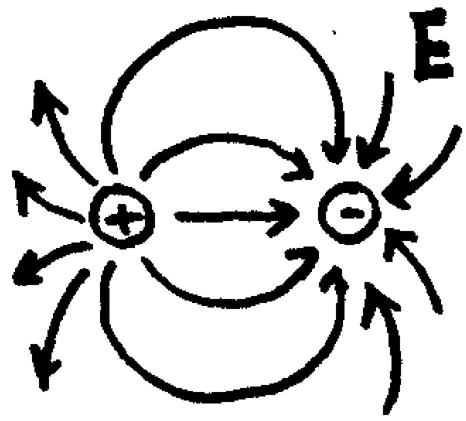
$$\therefore \int_V \left( \operatorname{div} \vec{\mathbf{E}} - \frac{\rho(\vec{\mathbf{r}})}{\varepsilon_o} \right) dV = 0$$

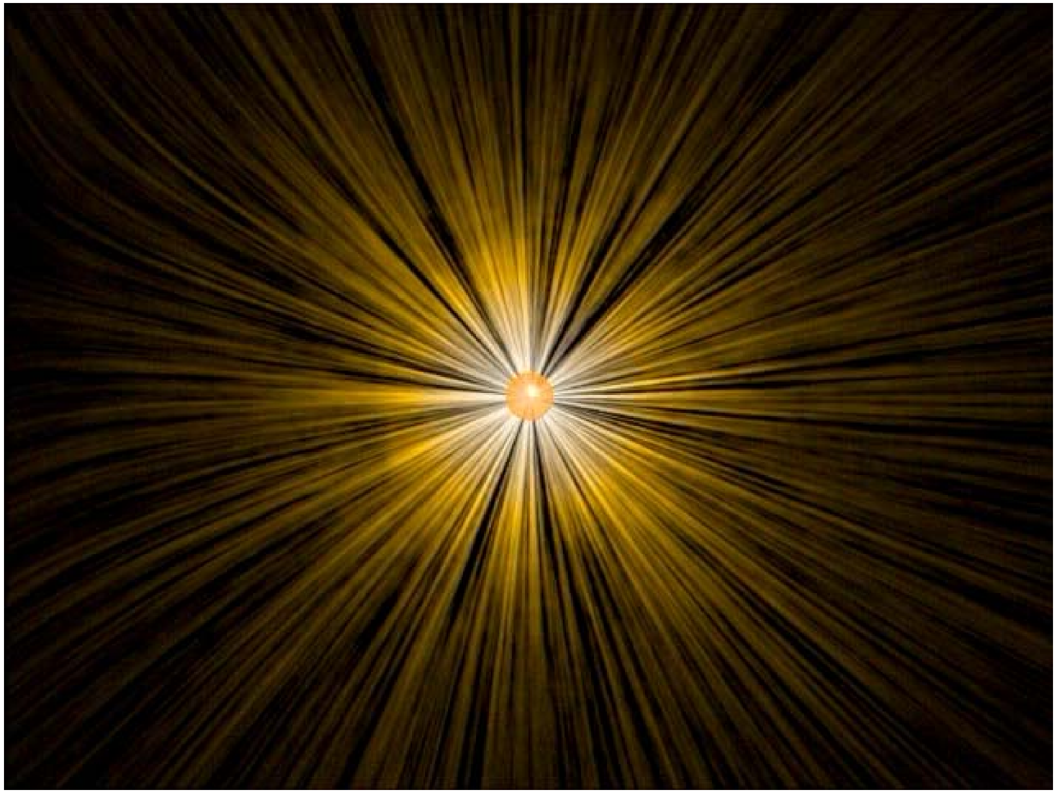
must be true for any volume $V$

$$\therefore \operatorname{div} \vec{\mathbf{E}} = \frac{\rho(\vec{\mathbf{r}})}{\varepsilon_o}$$

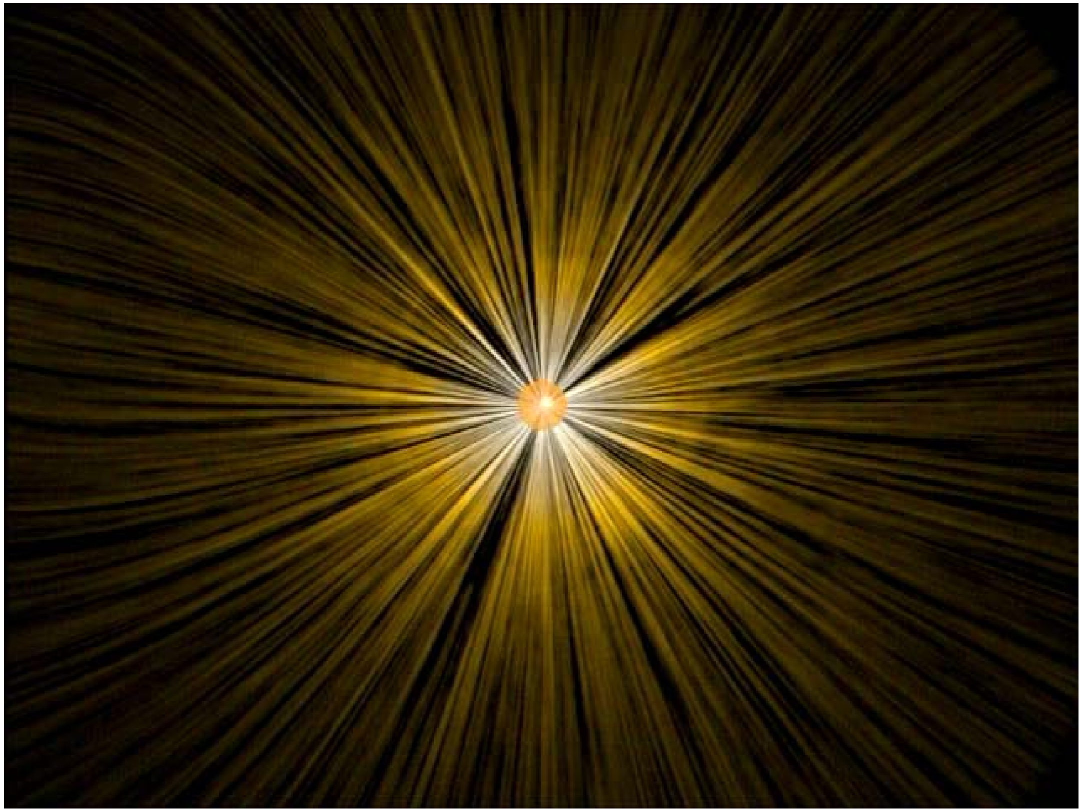
**Maxwell I**

Electric field diverges outwards from positive charge and inwards towards negative charge.





An animation of the motion of a positive charge moving past a massive charge which is also positive. The smaller charge is deflected away from the larger charge because of their mutual repulsion. This repulsion is primarily due to a pressure transmitted by the electric fields surrounding the charges.



An animation of the motion of a negative charge moving past a massive positive charge. The negative charge is deflected toward the positive charge because of the attraction between them. This attraction is primarily due to a tension transmitted by the electric fields surrounding the charges.

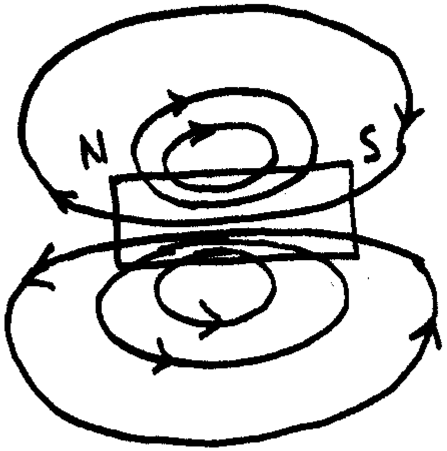
# Gauss law of magnetism

The net outward flux of any magnetic field across any closed surface $S$ is zero (there can be no magnetic monopoles).

$$\int_S \vec{\mathbf{B}} \bullet d\vec{\mathbf{a}} = 0$$

Then by the divergence theorem:

$$\text{div}\, \vec{\mathbf{B}} = 0 \qquad \textbf{Maxwell II}$$




Magnetic fields never diverge or converge.
They always form closed loops (circulate).

The animation shows two co-axial wire loops carrying current in the same sense. The loops attract one another. We show the field configuration here using the "iron filings" representation. The bottom wire loop carries three times the current of the top wire loop.

# Faraday's law of electromagnetic induction

$$\varepsilon_{induced} = -\frac{\partial \Phi_B}{\partial t} \qquad \text{where} \qquad \Phi_B = \int_S \vec{\mathbf{B}} \bullet d\vec{\mathbf{a}}$$

An emf can induced in a closed circuit by changing magnetic fields or by the motion of the circuit through a magnetic field. The circuit referred to could be a closed electric circuit or imaginary closed loop in space.

The potential difference induced along a small section $d\vec{\ell}$ of a circuit $C$ is given by $\vec{\mathbf{E}} \bullet d\vec{\ell}$ .

$$\therefore \varepsilon_{induced} = \oint_C \vec{\mathbf{E}} \bullet d\vec{\ell} = -\frac{\partial}{\partial t} \int_S \vec{\mathbf{B}} \bullet d\vec{\mathbf{a}}$$

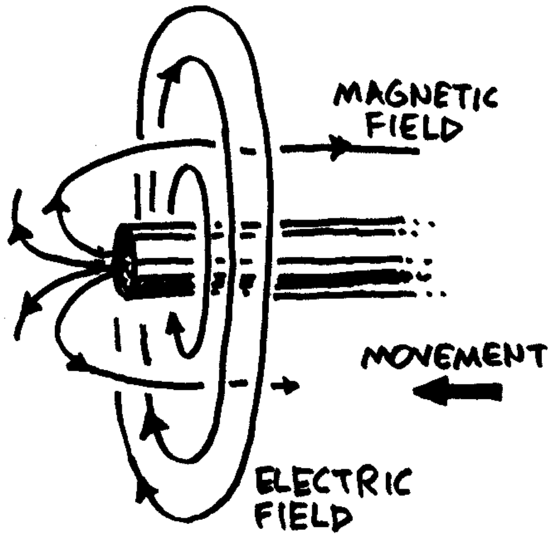
where the surface *S* is bounded by the circuit *C*.

It follows that induced electric fields are **non-conservative**.

# Faraday's law of electromagnetic induction ... 2

$$\oint_C \vec{\mathbf{E}} \bullet d\vec{\ell} = -\frac{\partial}{\partial t}\int_S \vec{\mathbf{B}} \bullet d\vec{\mathbf{a}}$$

By Stokes' theorem: $$\oint_C \vec{\mathbf{E}} \bullet d\vec{\ell} = \int_S \left(\vec{\nabla} \times \vec{\mathbf{E}}\right) \bullet d\vec{\mathbf{a}}$$




$$\therefore \int_S \left(\vec{\nabla} \times \vec{\mathbf{E}}\right) \bullet d\vec{\mathbf{a}} = -\frac{\partial}{\partial t}\int_S \vec{\mathbf{B}} \bullet d\vec{\mathbf{a}}$$

$$\therefore \int_S \left(\vec{\nabla} \times \vec{\mathbf{E}} + \frac{\partial \vec{\mathbf{B}}}{\partial t}\right) \bullet d\vec{\mathbf{a}} = 0$$

must be true for any circuit $C$

$$\therefore \text{ curl } \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$ **Maxwell III**

Changing magnetic fields induce electric fields which curl around the changing magnetic fields.

# Ampere's Law

Magnetic fields may be produced by electric currents:

$$\oint_C \vec{\mathbf{B}} \bullet d\vec{\ell} = \mu_o I$$

The line integral of a magnetic field $\vec{\mathbf{B}}$ around any closed path $C$ is equal to the permeability of free space $\mu_o$ multiplied by the current $I$ through the area enclosed by the path.

When the current is distributed continuously as a current density $\vec{\mathbf{j}}(\vec{\mathbf{r}})$ , then $I = \int_S \vec{\mathbf{j}} \bullet d\vec{\mathbf{a}}$

(The current $I$ across any surface $S$ bounded by the closed path $C$ is the flux of $\vec{\mathbf{j}}(\vec{\mathbf{r}})$ across $S$.)

Then $\oint_C \vec{\mathbf{B}} \bullet d\vec{\ell} = \mu_o \int_S \vec{\mathbf{j}} \bullet d\vec{\mathbf{a}}$

It follows that magnetic fields produced by electric currents are also **non-conservative**. Therefore the magnetic fields cannot be derived by scalar potentials, but only vector potentials.

# Ampere's Law ... 2

$$\oint_C \vec{\mathbf{B}} \bullet d\vec{\ell} = \mu_o \int_S \vec{\mathbf{j}} \bullet d\vec{\mathbf{a}}$$

By Stokes' theorem: $$\oint_C \vec{\mathbf{B}} \bullet d\vec{\ell} = \int_S \left(\vec{\nabla} \times \vec{\mathbf{B}}\right) \bullet d\vec{\mathbf{a}}$$

$$\therefore \int_S \left(\vec{\nabla} \times \vec{\mathbf{B}}\right) \bullet d\vec{\mathbf{a}} = \mu_o \int_S \vec{\mathbf{j}} \bullet d\vec{\mathbf{a}}$$

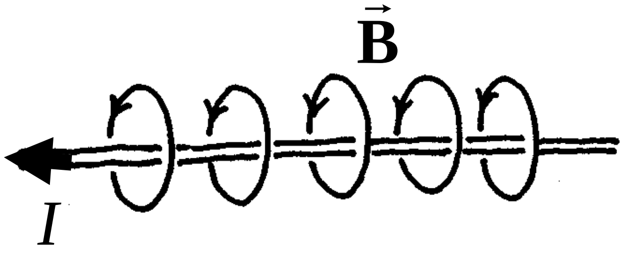
$$\therefore \int_S \left(\vec{\nabla} \times \vec{\mathbf{B}} - \mu_o \vec{\mathbf{j}}\right) \bullet d\vec{\mathbf{a}} = 0$$

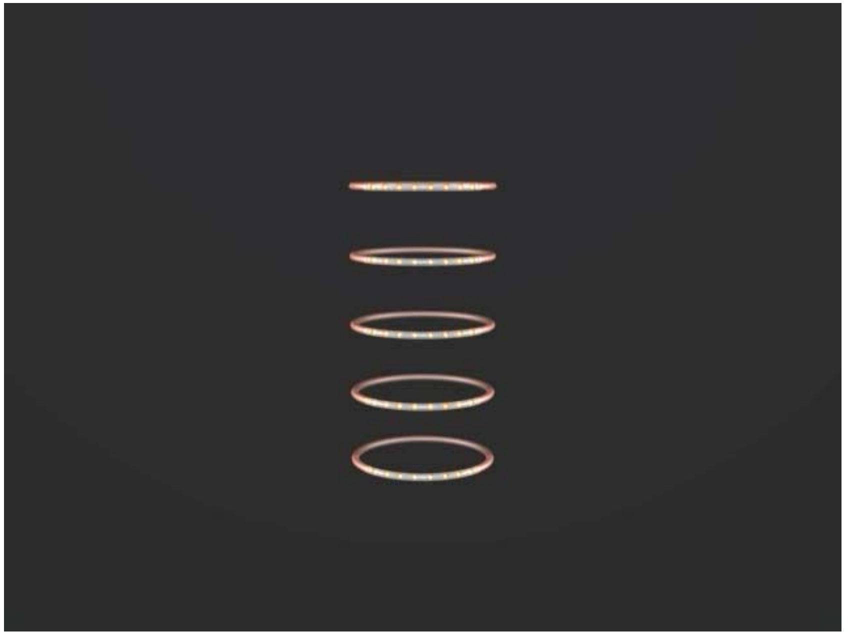
must be true for any circuit $C$

$$\therefore \text{curl } \vec{\mathbf{B}} = \mu_o \vec{\mathbf{j}}$$

**Maxwell IV (almost ...)**

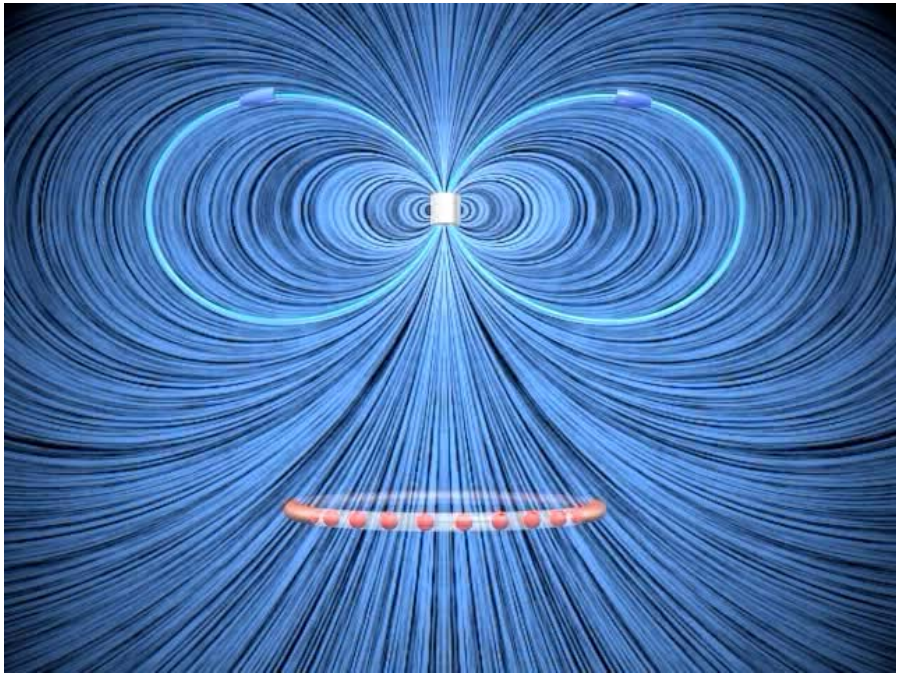
Magnetic field lines curl around electric currents (think of the magnetic field around a conducting wire).



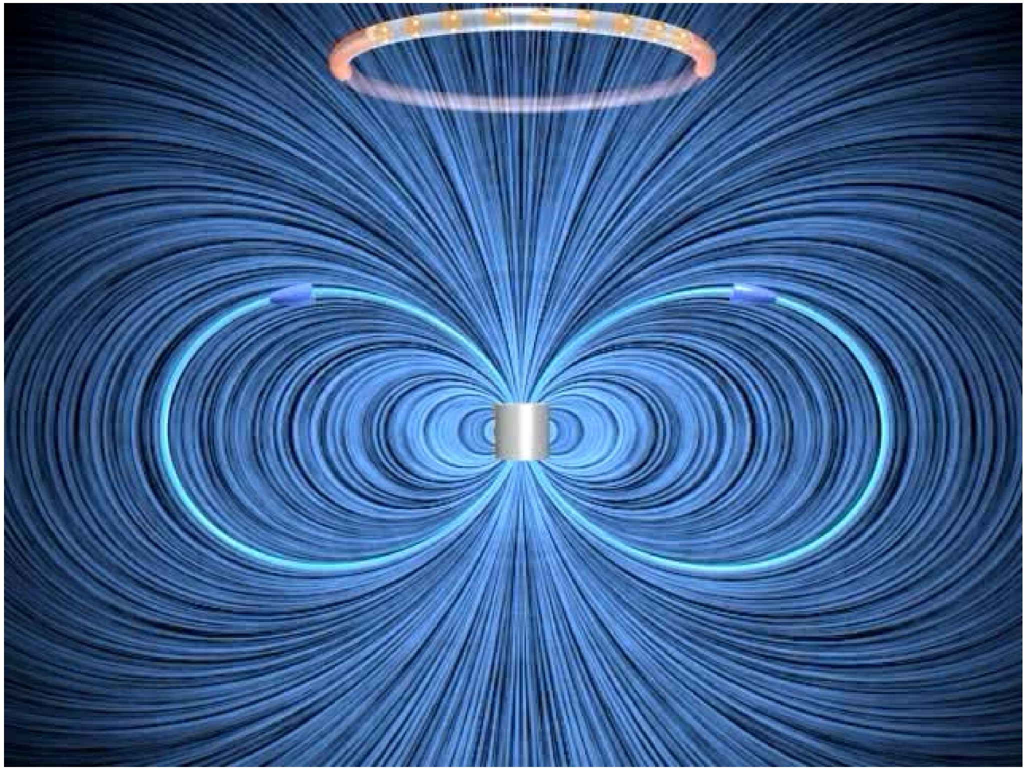


Suppose we have five rings that carry a number of free positive charges that are not moving. Since there is no current, there is no magnetic field. Now suppose a set of external agents come along (one for each charge) and simultaneously spin up the charges counterclockwise as seen from above, at the same time and at the same rate, in a manner that has been pre-arranged. Once the charges on the rings start to accelerate, there is a magnetic field in the space between the rings, mostly parallel to their common axis, which is stronger inside the rings than outside. This is the solenoid configuration.
As the magnetic flux through the rings grows, Faraday's Law tells us that there is an electric field induced by the time-changing magnetic field that is circulating clockwise as seen from above. The force on the charges due to this electric field is thus opposite the direction the external agents are trying to spin the rings up in (counterclockwise), and thus the agents have to do additional work to spin up the charges because of their charge. This is the source of the energy that is appearing in the magnetic field between the rings-the work done by the agents against the "back emf".

Over the time when the magnetic field is increasing in the animation, the agents moving the charges to a higher speed against the induced electric field are continually doing work. The electromagnetic energy that they are creating at the place where they are doing work (the path along which the charges move) flows both inward and outward. The direction of the flow of this energy is shown by the animated texture patterns. This is the electromagnetic energy flow that increases the strength of the magnetic field in the space between the rings as each positive charge is accelerated to a higher and higher velocity.



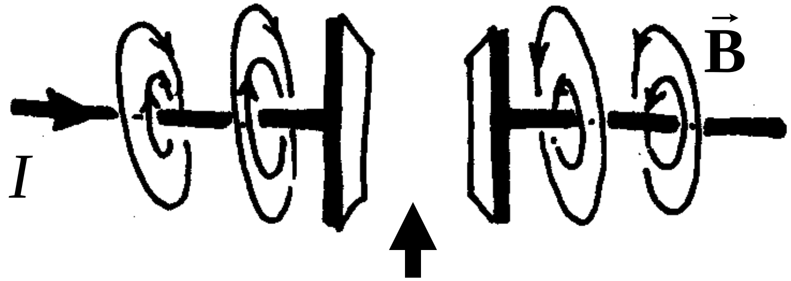
The animation shows the magnetic field configuration around a permanent magnet as it falls under gravity through a conducting non-magnet ring. The current in the ring is indicated by the small moving spheres. In this case, the magnet is light, the ring has zero resistance, and the magnet levitates above the ring. The motions of the field lines are in the direction of the local Poynting flux vector.



The animation shows the magnetic field configuration around a conducting non-magnetic ring as it falls under gravity in the magnetic field of a fixed permanent magnet. The current in the ring is indicated by the small moving spheres. In this case, the ring is heavy and has zero resistance, and falls past the magnet. The motions of the field lines are in the direction of the local Poynting flux vector.

The 4 “Maxwell” equations above express laws that came to Maxwell from other sources.  But Maxwell’s genius was to see that law IV was incomplete.

Consider a capacitor being charged. As the charge flows to the capacitor plates, a magnetic field rings the wire.
But what about between the plates?




No magnetic field here?

Does the magnetic field stop abruptly between the plates where there is no current?

Maxwell though not ...

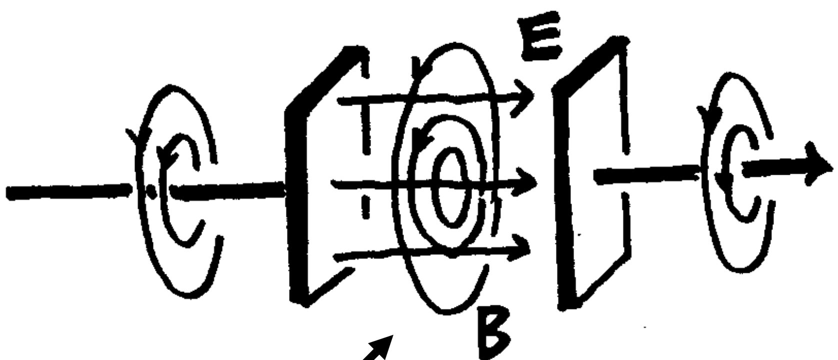
Maxwell reasoned that if changing magnetic fields induce electric fields (Faraday), then, symmetrically, changing electric fields might induce magnetic fields.

Although there was no experimental evidence for this at the time, Maxwell added an extra term to his fourth equation, saying that magnetic fields also curl around changing electric fields:

$$\text{curl}\ \vec{\mathbf{B}} = \mu_o \vec{\mathbf{j}} + \mu_o \varepsilon_o \frac{\partial \vec{\mathbf{E}}}{\partial t}$$

**Maxwell IV**

This term generates a magnetic field between the capacitor plates as the electric fields builds up.




Some years later, this magnetic field was detected ..!

# Maxwell's equations:

I $\text{div}\,\vec{\mathbf{E}} = \dfrac{\rho(\vec{\mathbf{r}})}{\varepsilon_o}$

II $\text{div}\,\vec{\mathbf{B}} = 0$

III $\text{curl}\,\vec{\mathbf{E}} = -\dfrac{\partial\vec{\mathbf{B}}}{\partial t}$

IV $\text{curl}\,\vec{\mathbf{B}} = \mu_o\vec{\mathbf{j}} + \mu_o\varepsilon_o\dfrac{\partial\vec{\mathbf{E}}}{\partial t}$

For **time independent** electric and magnetic fields:

I $\text{div}\,\vec{\mathbf{E}} = \dfrac{\rho(\vec{\mathbf{r}})}{\varepsilon_o}$

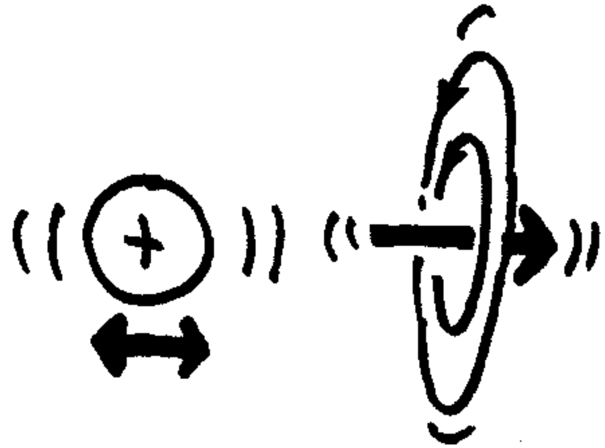
II $\text{div}\,\vec{\mathbf{B}} = 0$

III $\text{curl}\,\vec{\mathbf{E}} = 0$

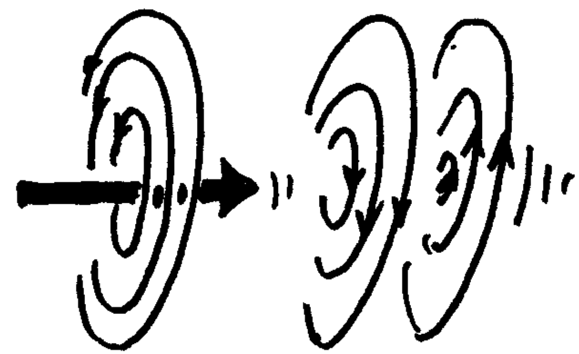
IV $\text{curl}\,\vec{\mathbf{B}} = \mu_o\vec{\mathbf{j}}$

The displacement current density term $\mu_o \varepsilon_o \frac{\partial \vec{\mathbf{E}}}{\partial t}$ in Maxwell's fourth equation led to a very large unexpected physics jackpot ...

Imagine a single electric charge being **vibrated**. In the space near the vibrating charge, the charge's electric field is changing, so it **induces** a magnetic field curling around it.




But the magnetic field is also changing, so it induces more electric field, which induces more magnetic field, etc, etc ...



The result is an **electromagnetic wave** of fields rippling out from the vibrating charge at the speed of light.

# Electromagnetic radiation continued ...

In empty space Maxwell III and IV become

$$\vec{\nabla}\times\vec{\mathbf{E}} = -\frac{\partial\vec{\mathbf{B}}}{\partial t} \qquad \text{and} \qquad \vec{\nabla}\times\vec{\mathbf{B}} = \mu_o\varepsilon_o\frac{\partial\vec{\mathbf{E}}}{\partial t}$$

$$\therefore \vec{\nabla}\times\left(\vec{\nabla}\times\vec{\mathbf{E}}\right) = -\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{\mathbf{B}}\right)$$

$$\therefore \vec{\nabla}\times\left(\vec{\nabla}\times\vec{\mathbf{E}}\right) = -\frac{\partial}{\partial t}\left(\mu_o\varepsilon_o\frac{\partial\vec{\mathbf{E}}}{\partial t}\right)$$

$$\therefore \vec{\nabla}\left(\vec{\nabla}\bullet\vec{\mathbf{E}}\right) - \nabla^2\vec{\mathbf{E}} = -\mu_o\varepsilon_o\frac{\partial^2\vec{\mathbf{E}}}{\partial t^2}$$

$$\therefore \nabla^2\vec{\mathbf{E}} = \mu_o\varepsilon_o\frac{\partial^2\vec{\mathbf{E}}}{\partial t^2}$$

$$\nabla^2 \vec{\mathbf{E}} = \mu_o \varepsilon_o \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2}$$

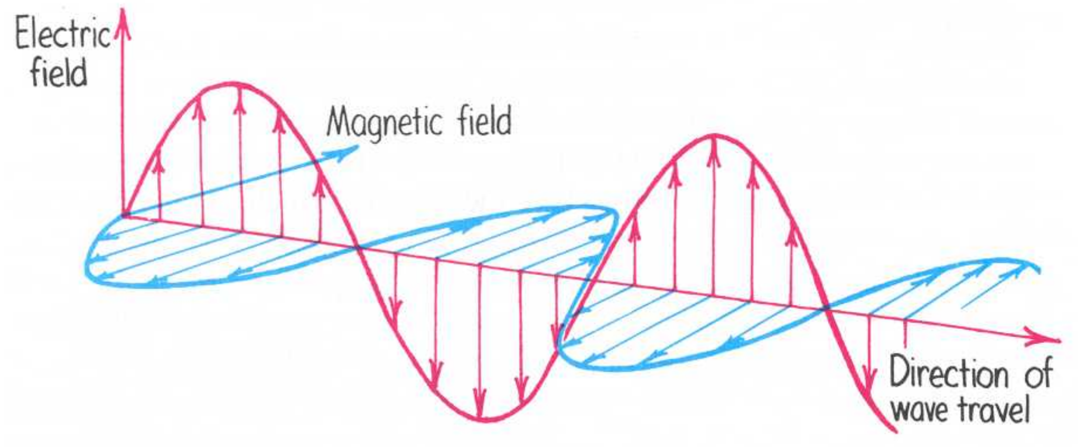
This partial differential has electric field solutions that represent waves propagating in empty space with speed

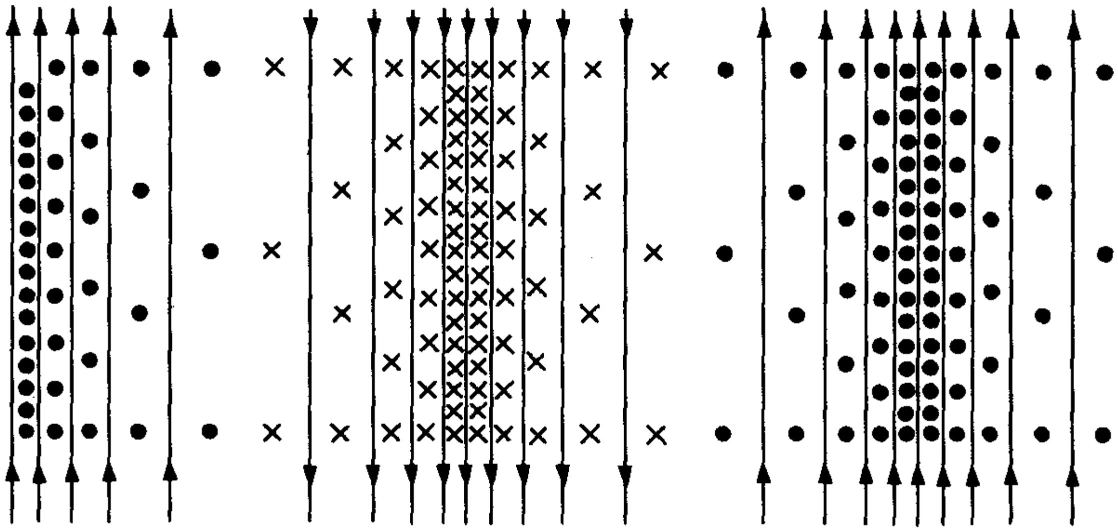
$$c = \frac{1}{\sqrt{\mu_o \varepsilon_o}} \quad = 2.998 \times 10^8 \text{ m s}^{-1}$$

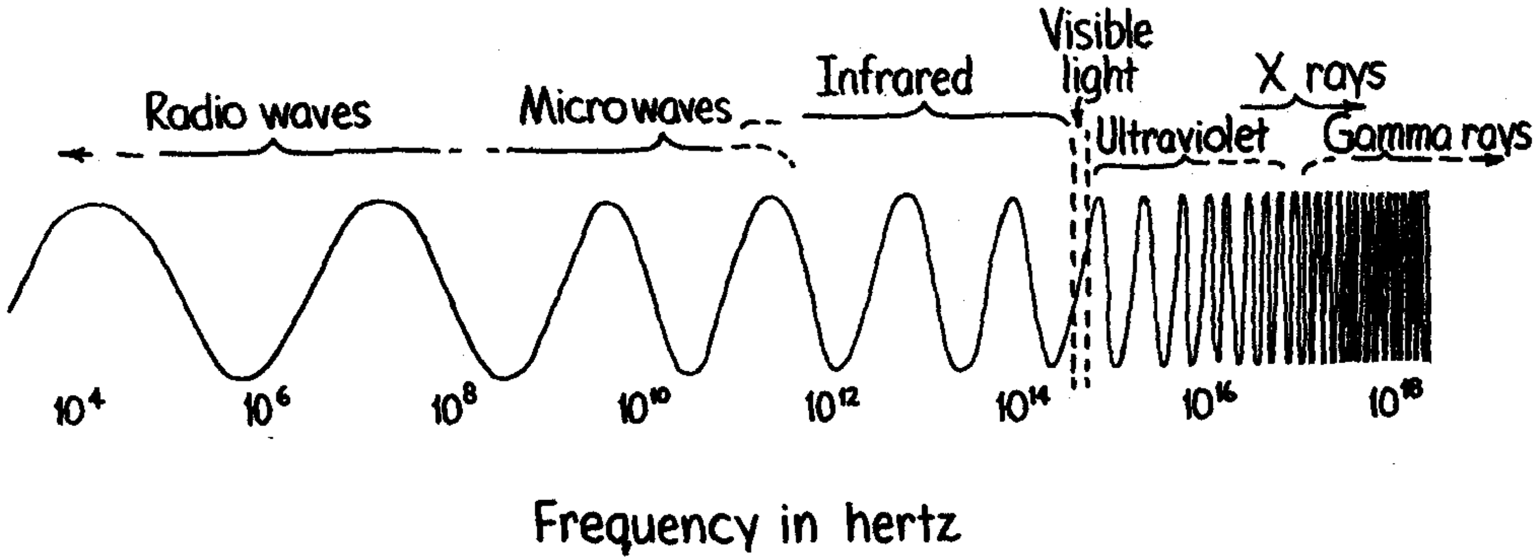
These waves together with associated magnetic field waves

$$\nabla^2 \vec{\mathbf{B}} = \mu_o \varepsilon_o \frac{\partial^2 \vec{\mathbf{B}}}{\partial t^2}$$

together describe constitute an **electromagnetic wave.**


Electric field
Magnetic field
Direction of wave travel




Radio waves
Microwaves
Infrared
Visible light
X rays
Ultraviolet
Gamma rays
10^4
10^6
10^8
10^10
10^12
10^14
10^16
10^18
Frequency in hertz

# The electrostatic scalar potential

For time independent fields $\vec{\nabla} \times \vec{\mathbf{E}} = 0$

showing that electrostatic field $\vec{\mathbf{E}}$ is a **conservative field.**

Since $\vec{\nabla} \times \left( \vec{\nabla} \varphi \right) = 0$ for any scalar field $\varphi$ .

we can write $\vec{\mathbf{E}} = -\vec{\nabla} V$

where the scalar field $V$ is called the **electrostatic potential**.

When the electric charge distribution is described by a charge density $\rho$ then we can use Gauss' Law $\vec{\nabla} \bullet \vec{\mathbf{E}} = \rho / \varepsilon_o$ to obtain:

$\vec{\nabla} \bullet \left( \vec{\nabla} V \right) = -\rho / \varepsilon_o$ or

| | |
|---|---|
| $\nabla^2 V = -\rho / \varepsilon_o$ | **Poisson's equation** |

If $\rho = 0$ everywhere, then

| | |
|---|---|
| $\nabla^2 V = 0$ | **Laplace's equation** |

# The magnetostatic vector potential

The sources of **magnetostatic** fields are steady current density distributions:

Amperes' law: $\vec{\nabla} \times \vec{\mathbf{B}} = \mu_o \vec{\mathbf{j}}$

i.e. $\vec{\mathbf{B}}$ is not conservative.

We also know that $\vec{\nabla} \bullet \vec{\mathbf{B}} = 0$ (no magnetic monopoles)

Since $\vec{\nabla} \bullet \left( \vec{\nabla} \times \vec{\mathbf{F}} \right) = 0$ for any vector field $\vec{\mathbf{F}}$ ,

we can write $\vec{\mathbf{B}} = \vec{\nabla} \times \vec{\mathbf{A}}$ for some $\vec{\mathbf{A}}$

where $\vec{\mathbf{A}}$ is the **magnetic vector potential**.

# The magnetostatic vector potential ...2

In order to define $\vec{\mathbf{A}}$ uniquely, we need to specify its divergence.

For time independent fields $\vec{\nabla} \bullet \vec{\mathbf{A}} = 0$ .

Therefore
$$\vec{\nabla} \times \vec{\mathbf{B}} = \vec{\nabla} \times \left( \vec{\nabla} \times \vec{\mathbf{A}} \right)$$
$$= \vec{\nabla} \left( \vec{\nabla} \bullet \vec{\mathbf{A}} \right) - \nabla^2 \vec{\mathbf{A}}$$
$$= -\nabla^2 \vec{\mathbf{A}} \qquad \text{since} \quad \vec{\nabla} \bullet \vec{\mathbf{A}} = 0$$

$$\therefore \quad \nabla^2 \vec{\mathbf{A}} = -\left( \vec{\nabla} \times \vec{\mathbf{B}} \right) = -\mu_o \vec{\mathbf{j}}$$

## Electrostatic scalar potential

Knowing $\rho$ and the boundary conditions for $V$

$$\nabla^2 V = -\rho/\varepsilon_o$$

$V$

$$\vec{\mathbf{E}} = -\vec{\nabla} V$$

$\vec{\mathbf{E}}$

## Magnetostatic vector potential

Knowing $\vec{\mathbf{j}}$ and the boundary conditions for $\vec{\mathbf{A}}$

$$\nabla^2 \vec{\mathbf{A}} = -\mu_o \vec{\mathbf{j}}$$

$\vec{\mathbf{A}}$

$$\vec{\mathbf{B}} = \vec{\nabla} \times \vec{\mathbf{A}}$$

$\vec{\mathbf{B}}$