PHY2013H
Tools and Skills component: Part B (30 lectures)

Rates and change and integration
Kinematics and ID motion
(Representing moving bodies)
Multiplication of vectors

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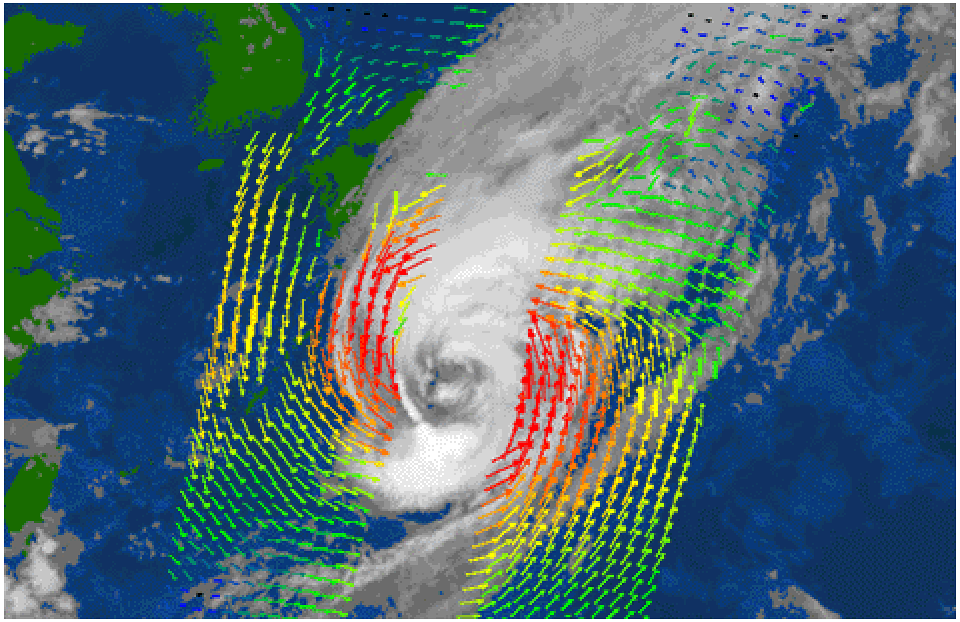
Notes, wps, etc will appear on vula.
Come to lectures on time, do the homework and wps, read the textbook, ask for help, keep up, focus in lectures, tutorial in Room L on Fridays, quick test on Thursdays, class work on other days, …



PHY1023H

Rates of change and integration

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Covering *Reese* pages 75 to 92, and some others on integration

The derivative of a function

The **derivative** of a function $f(x)$ is the function that describes the rate of change of $f(x)$ or the slope (gradient) of the curve of $f(x)$.

The "derivative of $f(x)$" $= f'(x) = \dfrac{df(x)}{dx} = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

For example if $f(x) = 3x^2$

then $$\frac{df(x)}{dx} = \frac{d(3x^2)}{dx} = \lim_{h \to 0} \frac{3(x+h)^2 - 3x^2}{h}$$

$$= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 3x^2}{h}$$

$$= \lim_{h \to 0} \frac{6xh + 3h^2}{h}$$

$$= \lim_{h \to 0} (6x + 3h) = 6x$$

There are many rules for differentiating different functions … you will meet these in your maths course.

In general, if $f(x) = ax^n$, where a is a constant and n is a real number, then

$$\frac{df(x)}{dx} = \frac{d}{dx}ax^n = nax^{n-1}$$

Exercises:

(a) $\frac{d}{dx}5x^3 =$

(b) $\frac{d}{dx}8x^{-2} =$

(c) $\frac{d}{dx}6x =$

(d) $\frac{d}{dx}7 =$

(e) $\frac{d}{dx}\left(5x^4 - 4x^2 + 3x^{-1}\right) =$

Average and instantaneous quantities

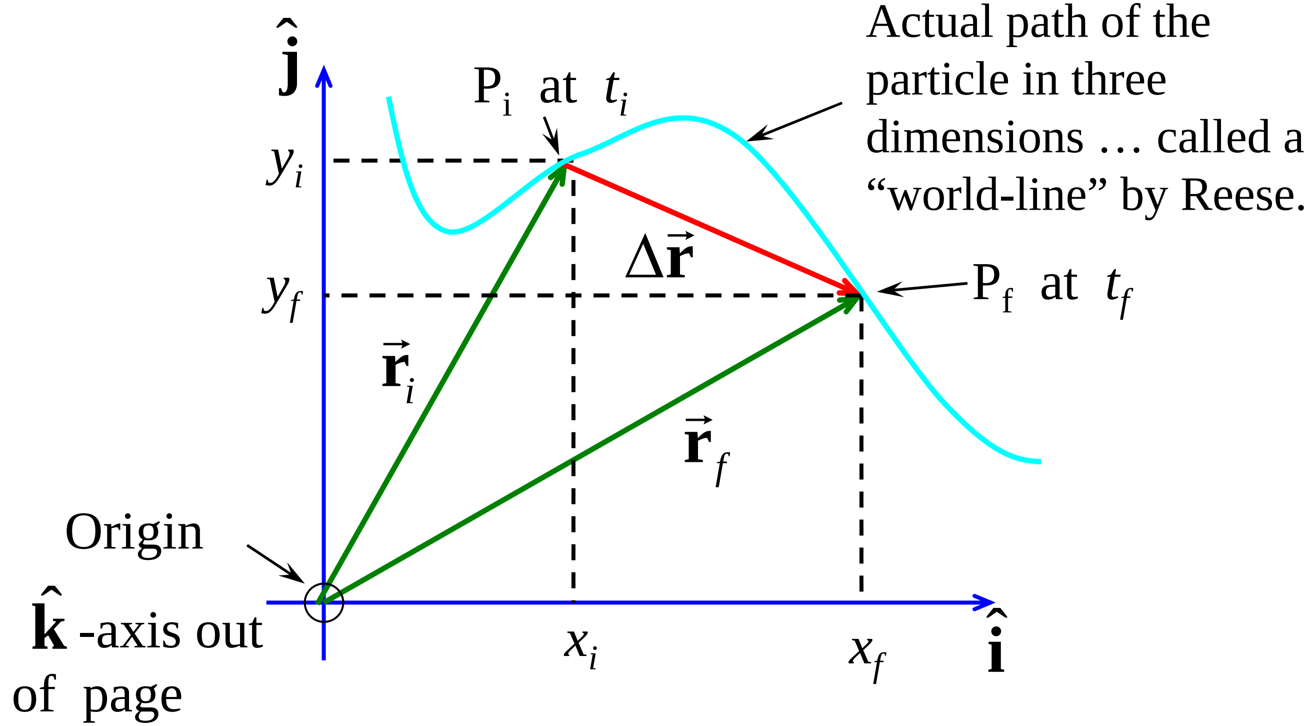
An **average quantity** tells us what has happened to a quantity over a time interval … i.e. how a quantity, such as velocity, changes over the time interval Δt.

An **instantaneous quantity** provides a value at an instant in time, which does not correspond to a time interval, even of very short duration.

… we will discuss these concepts in the contexts of velocity, acceleration and force. Unless clearly stated, most mathematical models in physics describe instantaneous quantities.

Position and displacement

(*Reese*: examples 3.1 ; 3.2 ; 3.3)



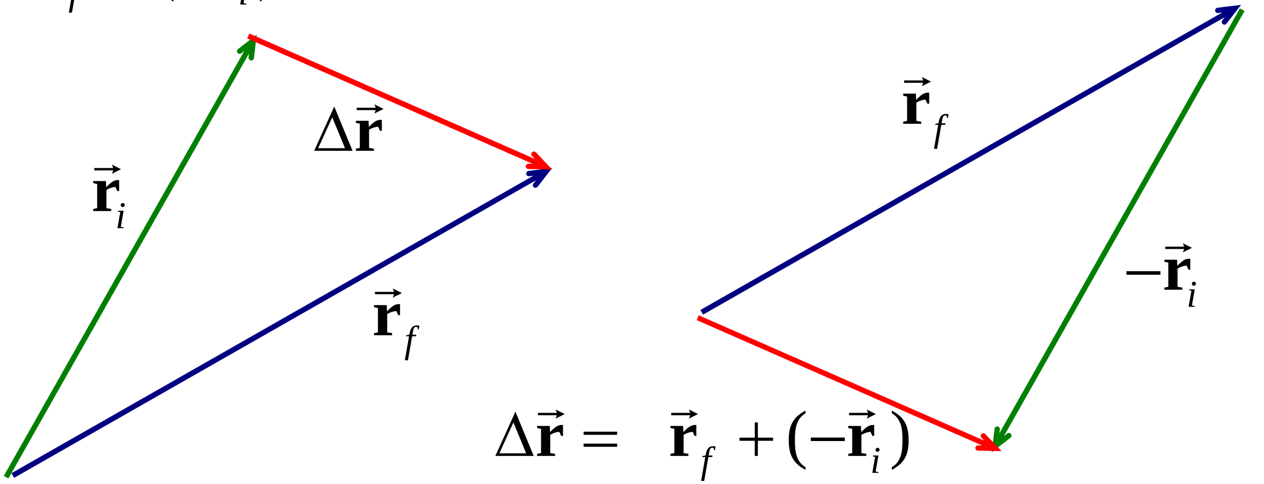
Consider a particle that follows the curved (blue) path in space. At time t_i it is at position P_i and time t_f it is at position P_f. To describe the motion of the particle, we use a three dimensional Cartesian coordinate system as shown.

Therefore at $t = t_i$, the particle is at position $\vec{\mathbf{r}}_i = x_i\hat{\mathbf{i}} + y_i\hat{\mathbf{j}} + z_i\hat{\mathbf{k}}$

and $t = t_f$, the particle is at position $\vec{\mathbf{r}}_f = x_f\hat{\mathbf{i}} + y_f\hat{\mathbf{j}} + z_f\hat{\mathbf{k}}$

The **displacement vector** is

$$\Delta\vec{\mathbf{r}} = \vec{\mathbf{r}}_f - \vec{\mathbf{r}}_i = \left(x_f - x_i\right)\hat{\mathbf{i}} + \left(y_f - y_i\right)\hat{\mathbf{j}} + \left(z_f - z_i\right)\hat{\mathbf{k}}$$
$$= \vec{\mathbf{r}}_f + (-\vec{\mathbf{r}}_i)$$



$$\Delta\vec{\mathbf{r}} = \vec{\mathbf{r}}_f + (-\vec{\mathbf{r}}_i)$$

Of course $\left|\Delta\vec{\mathbf{r}}\right|$ is <u>not</u> necessarily the distance from P_i to P_f.

In general an **instantaneous position vector** $\vec{\mathbf{r}}(t)$ describes the position of a particle at a particular instant in time relative to the origin of a set of coordinate axes:

$$\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$$

Clearly $\vec{\mathbf{r}}_f = \vec{\mathbf{r}}_i + \Delta\vec{\mathbf{r}}$

The velocity vector

The **average velocity** vector $$\vec{\mathbf{v}}_{av} = \frac{\Delta\vec{\mathbf{r}}}{\Delta t} = \frac{\vec{\mathbf{r}}_f - \vec{\mathbf{r}}_i}{t_f - t_i}$$

$$= \frac{\left(x_f - x_i\right)\hat{\mathbf{i}} + \left(y_f - y_i\right)\hat{\mathbf{j}} + \left(z_f - z_i\right)\hat{\mathbf{k}}}{t_f - t_i}$$

(*Reese*: examples 3.4 ; 3.5)

The **instantaneous velocity** vector $$\vec{\mathbf{v}}(t) = \frac{d\vec{\mathbf{r}}(t)}{dt}$$

$$= \frac{d}{dt}\left(x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}\right)$$

$$= \frac{dx(t)}{dt}\hat{\mathbf{i}} + \frac{dy(t)}{dt}\hat{\mathbf{j}} + \frac{dz(t)}{dt}\hat{\mathbf{k}}$$

(*Reese*: example 3.7)

***Reese* strategic example 3.6 page 82**

You drive your car 2.00 km down a drag strip, then 2.50 km in the opposite direction, completing the excursion in 180 s.

a. Choose an appropriate coordinate system to describe the motion.
b. Determine the initial and final position vectors and the change in the position vector during the time interval.
c. Find the average velocity.
d. Find the average speed.
e. Explain why the average speed of the car is not equal to the magnitude of the average velocity.

***Reese* strategic example 3.8 page 85**

The one-dimensional position vector of a sports car is given by

$$\vec{\mathbf{r}}(t) = [4.00\ \text{m} - (2.00\ \text{m s}^{-1})t + (3.00\ \text{m s}^{-2})t^2]\hat{\mathbf{i}}$$

a. Make an x versus t graph including at least the interval from t = 0 s to t = 3.00 s.

b. Find the velocity $\vec{\mathbf{v}}(t)$.

c. Specify the Cartesian component $v_x(t)$ of the velocity at any instant t.

d. Make a graph of $v_x(t)$ versus t.

e. Find the velocity it the instant when t = 2.00 s.

f. Determine at what time the velocity is zero.

g. Describe the motion of the particle.

The acceleration vector

The **average acceleration** vector $\vec{\mathbf{a}}_{av} = \dfrac{\Delta\vec{\mathbf{v}}}{\Delta t} = \dfrac{\vec{\mathbf{v}}_f - \vec{\mathbf{v}}_i}{t_f - t_i}$

The **instantaneous acceleration** vector

$$\vec{\mathbf{a}}(t) = \frac{d\vec{\mathbf{v}}(t)}{dt}$$

$$= \frac{dv_x(t)}{dt}\hat{\mathbf{i}} + \frac{dv_y(t)}{dt}\hat{\mathbf{j}} + \frac{dv_z(t)}{dt}\hat{\mathbf{k}}$$

$$= \frac{d^2x(t)}{dt^2}\hat{\mathbf{i}} + \frac{d^2y(t)}{dt^2}\hat{\mathbf{j}} + \frac{d^2z(t)}{dt^2}\hat{\mathbf{k}}$$

(*Reese*: examples 3.9 ; 3.10)

Force as the rate of change of momentum

The **average force** vector

$$\vec{\mathbf{F}}_{av} = \frac{\Delta\vec{\mathbf{p}}}{\Delta t} = \frac{\vec{\mathbf{p}}_f - \vec{\mathbf{p}}_i}{t_f - t_i} = \frac{\vec{\mathbf{p}}(t_f) - \vec{\mathbf{p}}(t_i)}{t_f - t_i}$$

The **instantaneous force** vector

$$\vec{\mathbf{F}}(t) = \lim_{\Delta t \to 0} \frac{\Delta\vec{\mathbf{p}}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{\mathbf{p}}(t+\Delta t) - \vec{\mathbf{p}}(t)}{\Delta t} = \frac{d\vec{\mathbf{p}}(t)}{dt}$$

$$= \frac{d}{dt} m(t)\vec{\mathbf{v}}(t)$$

$$= m(t)\frac{d\vec{\mathbf{v}}(t)}{dt} + \vec{\mathbf{v}}(t)\frac{dm(t)}{dt}$$

$$= m(t)\vec{\mathbf{a}}(t) + \vec{\mathbf{v}}(t)\frac{dm(t)}{dt}$$

(Newton’s Second Law)

Example 1

The position of a 3 kg object as a function of time is given by:

$$\vec{\mathbf{r}}(t) = (2t^2 + 1)\hat{\mathbf{i}} + (5t + 2)\hat{\mathbf{j}} + (6t - 5t^2)\hat{\mathbf{k}} \quad \text{metres}$$

(a) Determine the position of the object at t = 1 s.
(b) Determine the displacement of the object between t = 1 s and t = 3 s.
(c) Determine the velocity of the object at t = 3 s.
(d) Determine the average velocity of the object between t = 1 s and t = 3 s.
(e) Determine the acceleration of the object at t = 5 s.
(f) Determine the average acceleration of the object between t = 1 s and t = 3 s.

Example 2

Bugs walks 3 metres in the $\hat{\mathbf{i}}$-direction for 6 seconds, and then 6 metres in a direction –30° to the $\hat{\mathbf{i}}$-direction for another 6 seconds. What is his average velocity?

Answer: $0.68\,\hat{\mathbf{i}} - 0.25\,\hat{\mathbf{j}}$ m s^{-1}

Example 3

At time t = 10 s, an object is moving with a velocity of 20 m s^{-1}, 30° clockwise from the $\hat{\mathbf{i}}$-axis. At t = 80 s, it is moving with a velocity of 45 m s^{-1}, 10° anticlockwise from the $\hat{\mathbf{j}}$-axis.
What is the average acceleration of the object for the whole journey?

Answer: $-0.36\,\hat{\mathbf{i}} + 0.78\,\hat{\mathbf{j}}$ m s^{-2}

Example 4

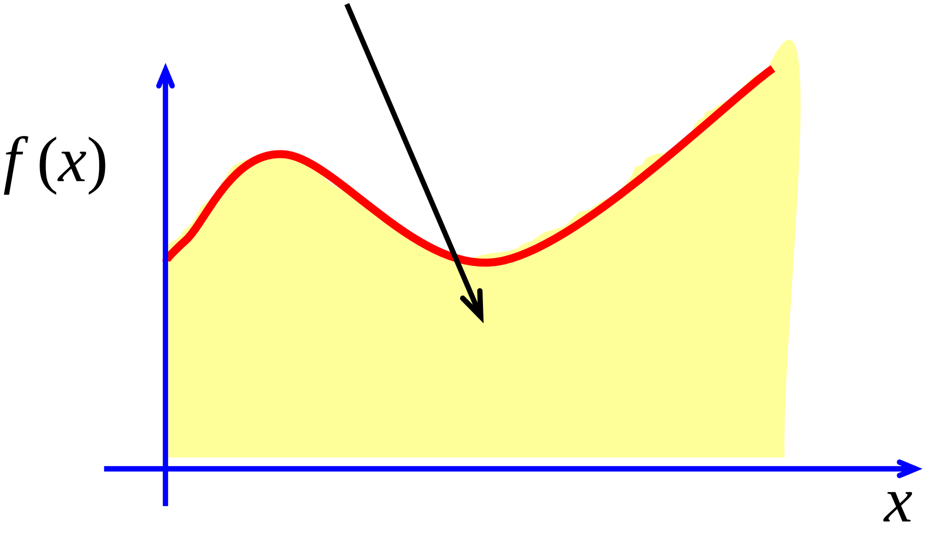
At time t = 5 s, you are at 25 m, 20° anticlockwise from the $\hat{\mathbf{i}}$-axis. At t = 25 s you are standing at a position 10 $\hat{\mathbf{j}}$ m. What is your average velocity for the whole journey? Show clearly how you arrived at your answer.

Answer: $-1.17\,\hat{\mathbf{i}} + 0.07\,\hat{\mathbf{j}}$ m s^{-1}

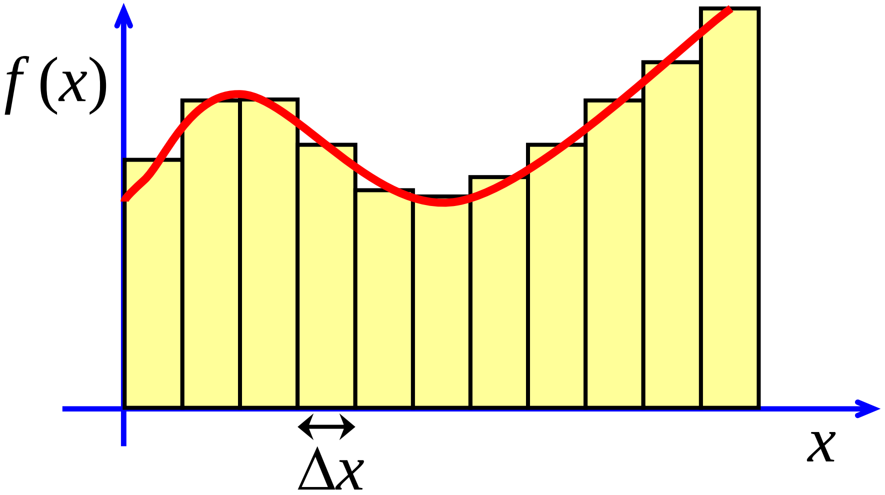
Integration in physics

Basically, integration is the inverse operation of differentiation, and is therefore sometimes referred to as anti-differentiation.

Consider a function $f(x)$ as shown below.
How can we find the area under such a curve?

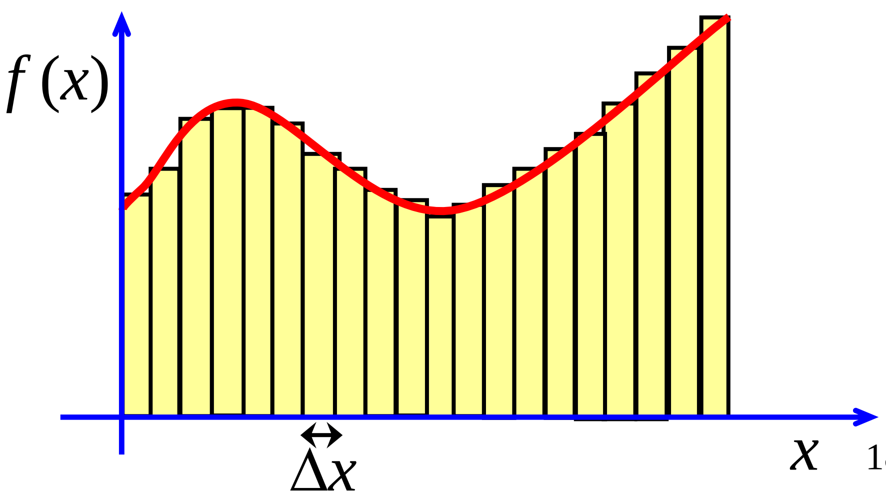


One way would be to construct thin, equally wide rectangles under the curve, as shown.



If each rectangle has width Δx , then the area under the curve $f(x)$ is approximated by $\sum f_i(x_i)\Delta x$

As we use rectangles with smaller Δx , $\sum f_i(x_i)\Delta x$ approaches the area under the curve …



Of course, we would try to make these rectangles as small as possible, infinitesimally small if we could.
The operation of **integration** does precisely that for you …

The "integral of $f(x)$" = $\int f(x)dx$

$= A(x)$ = area under the function $f(x)$.

This is true if $\dfrac{dA(x)}{dx} = f(x)$

… therefore $A(x)$ is called the anti-derivative of $f(x)$.

$A(x)$ is not a unique function since …

$$\frac{d}{dx}(A(x)+C) = \frac{d}{dx}A(x) = f(x) \quad \text{for } \underline{\text{any constant}}\ C$$

$\int f(x)dx$ is called an **indefinite integral** since it provides the function which generally describes the "area" under the function $f(x)$. Note that the "area" can be positive or negative.

Note that there are many, many more complicated cases of integration which you will learn about in your mathematics course. For now we are only interested in dealing with the case

$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + C$$

Exercises:

(a) $\int 4x dx =$

(b) $\int (9x^2 + 16x^3) dx =$

(c) $\int \frac{x^4}{2} dx =$

(d) $\int x^{-3} dx =$

If we are only interested in the area between two particular x values, then we write

$$\int_{x_i}^{x_f} f(x)dx$$

where x_f and x_i are called the **limits** of the **definite integral**. We interpret this mathematical statement as "determine the area under the function $f(x)$ between $x = x_i$ and $x = x_f$."

A definite integral may result in a positive number, a negative number or even zero as the result.

For example… $$\int_1^2 \left(x^3+3\right)dx = \left.\frac{x^4}{4}+3x+C\right|_1^2$$

$$= \left(\frac{(2)^4}{4}+3(2)+C\right)-\left(\frac{(1)^4}{4}+3(1)+C\right)$$

$$= 4+6-0.25-3 \quad = \quad 6.75$$

Exercises:

(a) $$\int_{-1}^{1} 4x\,dx =$$

(b) $$\int_{1}^{3} (3x^2 - 4x + 8)\,dx =$$

An example from physics: Work and energy

The work done W by a force $\vec{\mathbf{F}}(\vec{\mathbf{r}})$ in moving a particle from point a to point b is defined as

$$W = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}) \bullet d\vec{\mathbf{r}}$$

"dot product" … see later

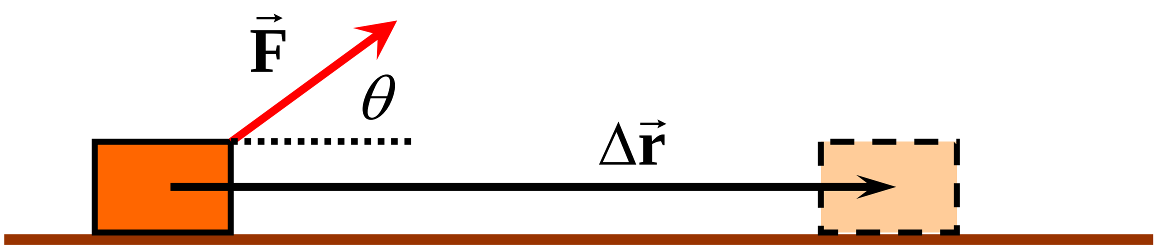
Unit of work: joule (J)

Work done is a **scalar** quantity.

2 special cases:

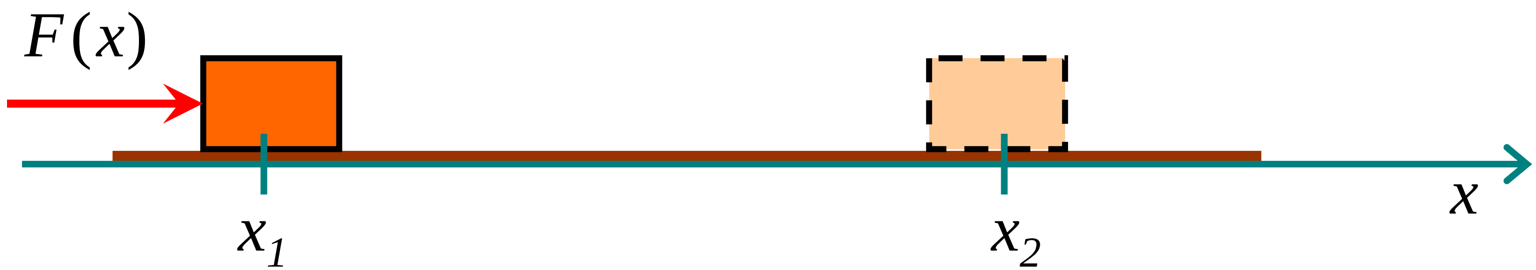
(i) If the force is not variable, i.e. $\vec{\mathbf{F}}$ is constant, and $\Delta\vec{\mathbf{r}}$ is in a straight line, then

$$W = \vec{\mathbf{F}} \bullet \Delta\vec{\mathbf{r}} = \left|\vec{\mathbf{F}}\right|\left|\Delta\vec{\mathbf{r}}\right| \cos\theta = F\,\Delta r \cos\theta$$



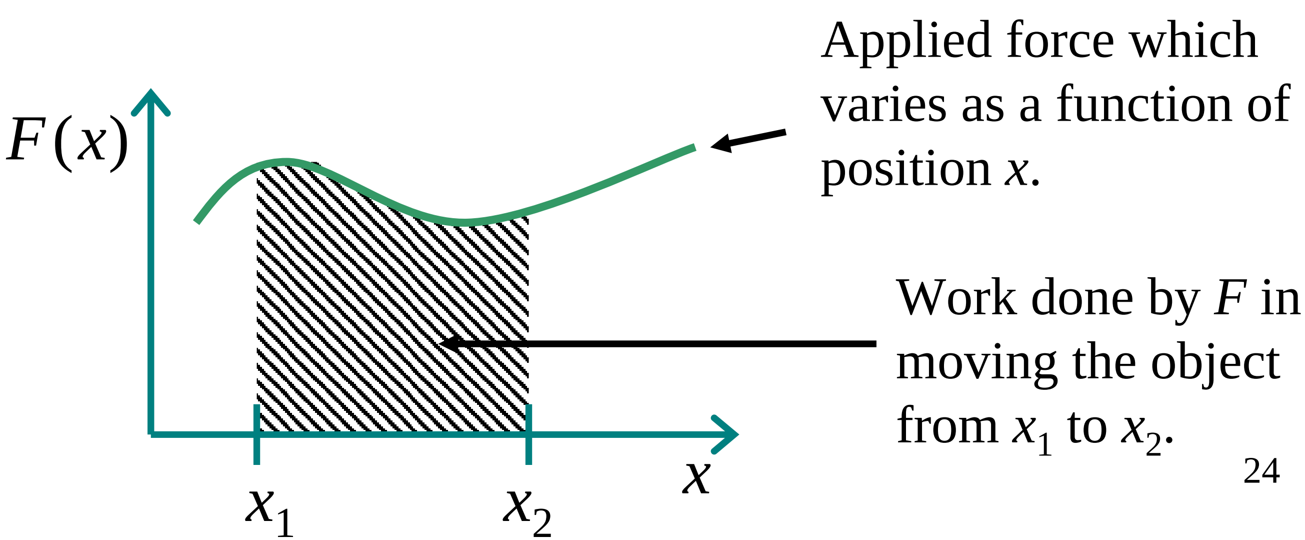
(ii) If $\vec{\mathbf{F}}$ is variable, but the displacement is still in only one direction,

then $W = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}) \bullet d\vec{\mathbf{r}}$ simplifies to $W = \int_{x_1}^{x_2} F(x)dx$

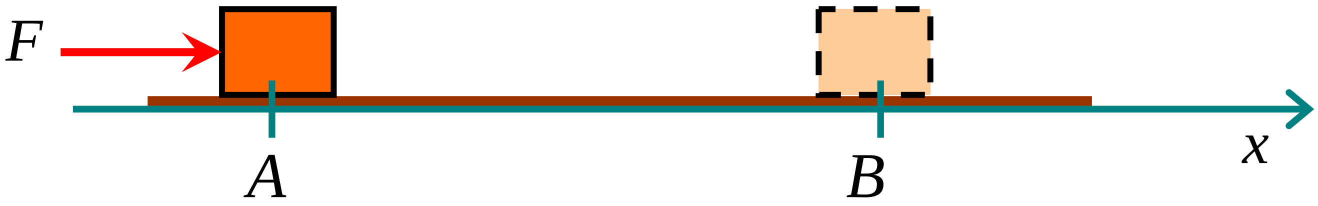


This integral may be interpreted as the area under the $F(x)$ - x curve.

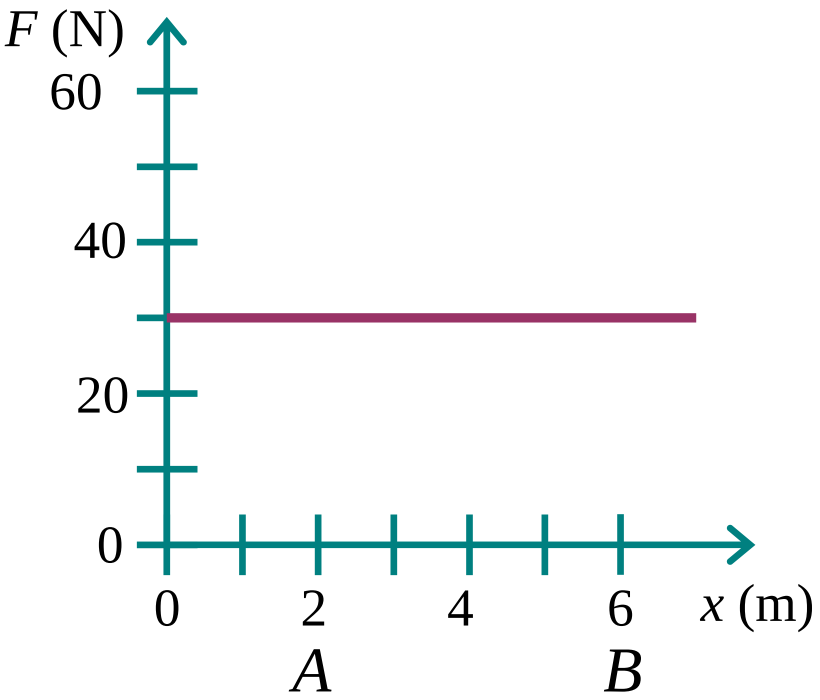
For example:



Example 1

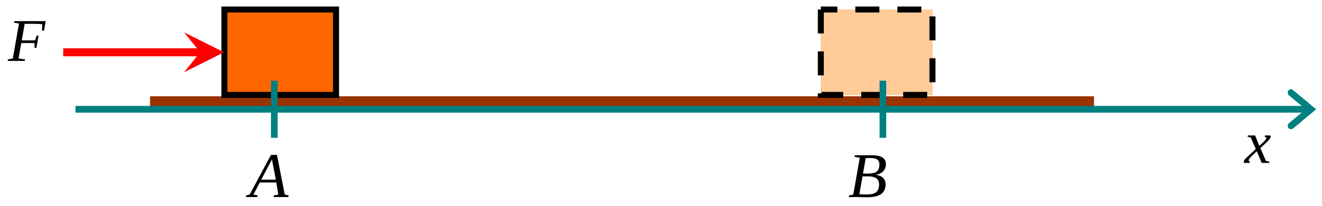


Shown alongside is a force-distance graph for a constant force acting on a mass in the *x*-direction as shown in the diagram. From the graph, calculate how much work is done by this force in moving the mass from point A to point B.

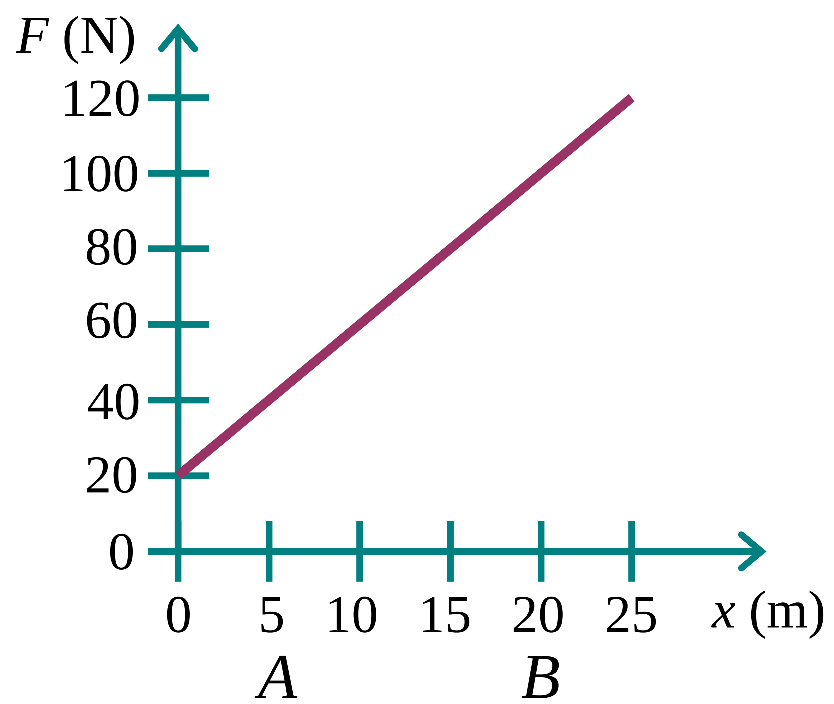


Example 2

A force *F* moves a box from point A to point B along the *x*-axis.



Shown alongside is the force-distance graph for this situation.



(a) From the graph, calculate the work done by this force in moving the box from A to B.
(b) Using an appropriate integral, also calculate this work.

Answer: 1050 J

Example 3

The force acting on an object along the x-axis is described by the function

$$F(x) = 3x + 5 \text{ newtons.}$$

(a) Sketch the graph of $F(x)$ versus x.

(b) Using the graph, calculate the work done by the force to move the object from $x = 2$ m to $x = 5$ m.

(c) Calculate the work done in moving the object from $x = 2$ m to $x = 5$ m by using an integral.
Compare your answer to part (b).

(d) Explain carefully the physical meaning of the integral.

Answer: 46.5 J

Another context from physics … the work done by a gas

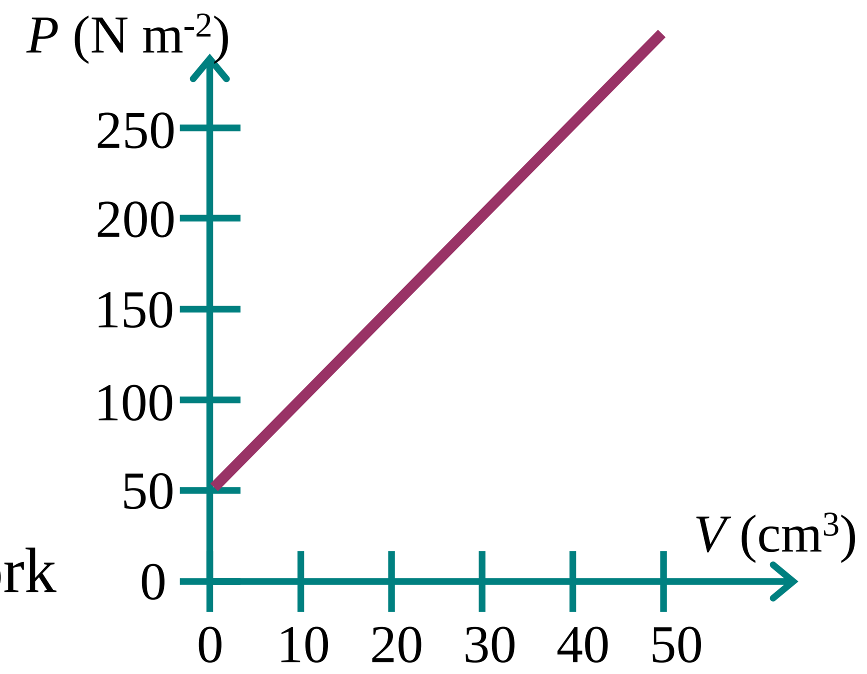
The work done by a gas is described by $$W = \int_{V_1}^{V_2} P(V)\,dV$$

… where $P(V)$ is the pressure as a function of the volume V. Therefore the work done by a gas is given by the area under a $P(V)$ versus V curve.

For example: The pressure (P) of a gas is related to its volume (V) graph as shown:

(a) Use the graph to determine the work done by the gas when the volume is increased from 10 cm^3 to 40 cm^3.

(b) Use an integral to calculate the work done by the gas when the volume is increased from 10 cm^3 to 40 cm^3.



Answer: 5.25×10^{-3} J

Returning to linear momentum …

Linear momentum for a single particle: $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$

Recall that Newton II can be expressed as $\vec{\mathbf{F}}_{ext} = \dfrac{d\vec{\mathbf{p}}}{dt}$

For a system of particles:

$$\vec{\mathbf{p}} = \sum_i \vec{\mathbf{p}}_i = m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2 + \ldots + m_n\vec{\mathbf{v}}_n = \sum_i m_i\vec{\mathbf{v}}_i = M\vec{\mathbf{v}}_{cm}$$

Newton II for a system of particles:

$$\frac{d\vec{\mathbf{p}}}{dt} = M\frac{d\vec{\mathbf{v}}_{cm}}{dt} = M\vec{\mathbf{a}}_{cm} = \vec{\mathbf{F}}_{ext}$$

If the external force on a system of particles is zero $\vec{\mathbf{F}}_{ext} = 0$,

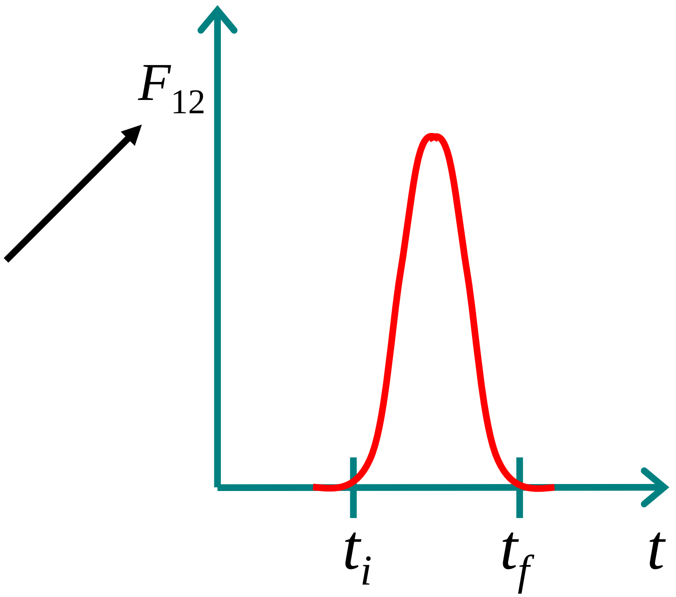
then $\dfrac{d\vec{\mathbf{p}}}{dt} = 0$ or $\vec{\mathbf{p}} =$ constant

If there is no resultant external force acting on an isolated system, then the total linear momentum remains constant.

Impulse and momentum

A collision is when two bodies interact over a short time interval. The forces that the bodies exert on each other are usually so strong during the collision that all forces acting on a body may be ignored.

During a collision between two bodies (1 and 2), the contact force exerted by one body on the other jumps from zero to a very large value and then abruptly drops to zero again.

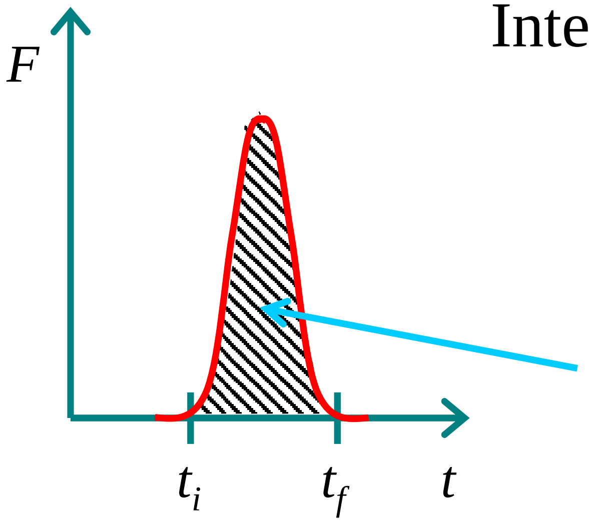


The time interval $\Delta t = t_f - t_i$ is usually very small.

Note that $\vec{\mathbf{F}}_{12} = -\vec{\mathbf{F}}_{21}$ for the collision

Impulse and momentum ... 2

$\vec{\mathbf{F}} = \dfrac{d\vec{\mathbf{p}}}{dt}$ for each object $\rightarrow \vec{\mathbf{F}}dt = d\vec{\mathbf{p}}$

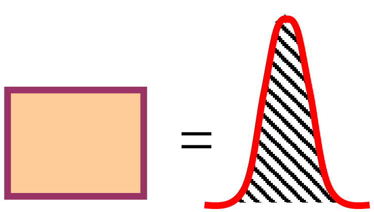


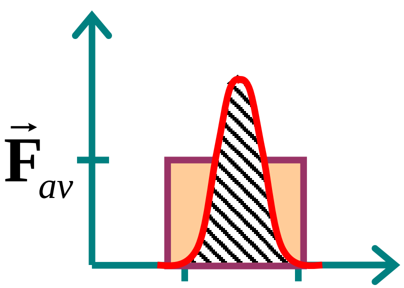
Integrating over the time of the collision:

$$\underbrace{\int_{t_i}^{t_f} \vec{\mathbf{F}}dt}_{} = \int_{p_i}^{p_f} d\vec{\mathbf{p}}$$

Impulse $\vec{\mathbf{J}} = \vec{\mathbf{p}}_f - \vec{\mathbf{p}}_i = \Delta\vec{\mathbf{p}}$

Sometimes it is useful to use the average force $\vec{\mathbf{F}}_{av}$ acting for time Δt to give the same impulse $\vec{\mathbf{J}}$ and $\Delta\vec{\mathbf{p}}$.





Then $\vec{\mathbf{F}}_{av}\Delta t = \vec{\mathbf{J}} = \Delta\vec{\mathbf{p}}$

Example

An object of mass 2 kg has a velocity of

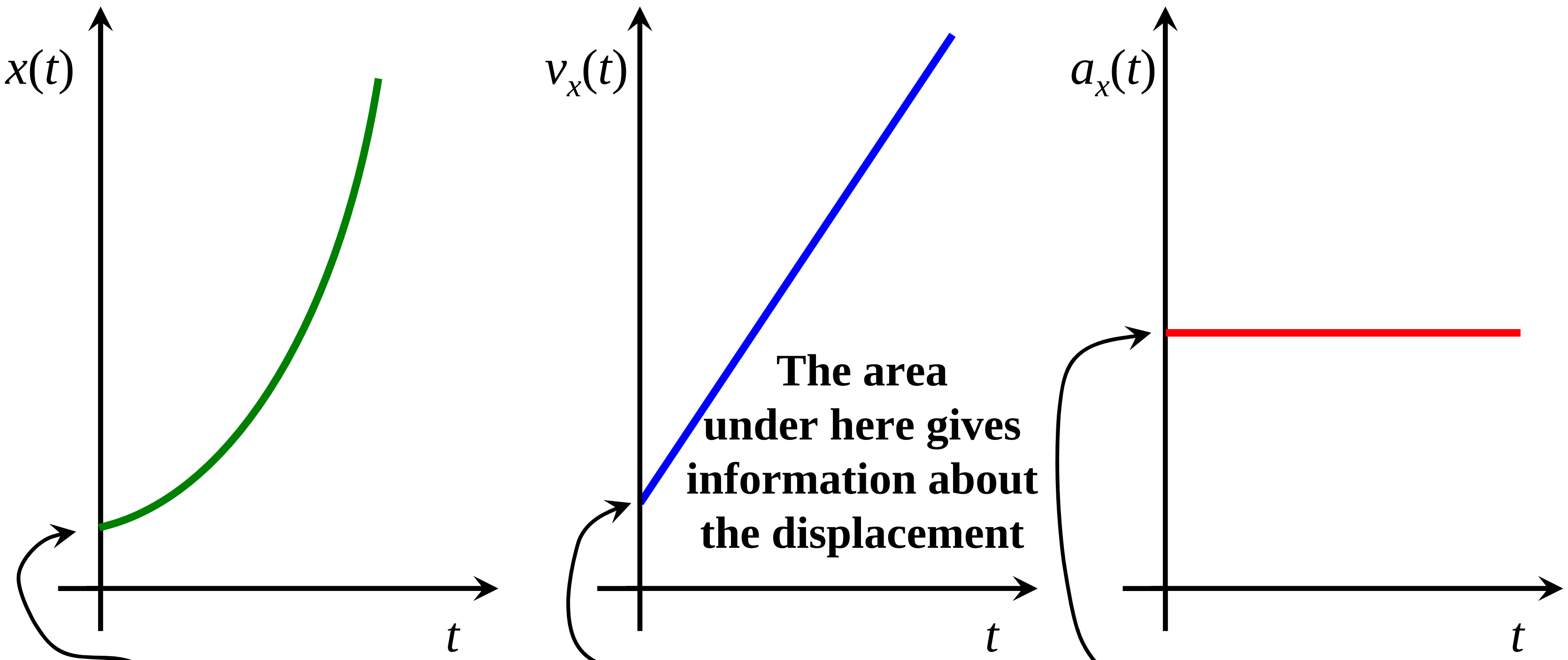
$$\vec{\mathbf{v}}(t) \;=\; (2t+1)\hat{\mathbf{i}} \;\; \text{m s}^{-1}$$

(a) Calculate the momentum of the object at t = 1 s.
(b) Calculate the force acting on the object at t = 2 s.
(c) Calculate the average force acting on the object between t = 3 s and t = 5 s.
(d) Sketch a graph of the momentum of the object as a function of time.
(e) From your graph, determine:

(i) the force acting on the object at t = 2 s
(ii) the average force acting on the object between t = 3 s and t = 5 s.

Using graphs of motion … coming later

Graphical representations of constant accelerated motion



$$x(t) = x_o + u_x t + \tfrac{1}{2}a_x t^2$$

$$\frac{d}{dt}x(t) = \frac{d}{dt}\left(x_o + u_x t + \tfrac{1}{2}a_x t^2\right) \rightarrow v_x(t) = u_x + a_x t$$

$$\frac{d}{dt}v_x(t) = \frac{d}{dt}\left(u_x + a_x t\right) \rightarrow a_x = \text{constant}$$