PHY123H

Mechanics Part C

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... see Chapters 8, 9 & 10 in
University Physics
by Ronald Reese

Work and Energy

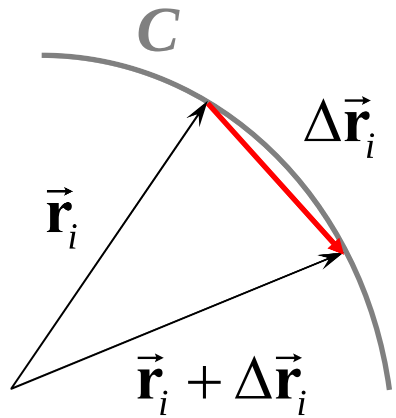
The work done W by a force $\vec{\mathbf{F}}(\vec{\mathbf{r}})$ in moving a particle from point *a* to point *b* is defined as

$$W = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}) \bullet d\vec{\mathbf{r}}$$

Unit of work: joule (J)

Work done is a **scalar** quantity.

Consider, for example, a comet moving along a path around the sun and let C be the segment of this path.



$\vec{\mathbf{F}}(\vec{\mathbf{r}})$ is the sun's gravitational force acting on the comet at position $\vec{\mathbf{r}}$.
Both the magnitude <u>and</u> direction of $\vec{\mathbf{F}}(\vec{\mathbf{r}})$ varies with $\vec{\mathbf{r}}$.

The work done by $\vec{\mathbf{F}}(\vec{\mathbf{r}})$ along the path segment C is a scalar quantity.

For the small path segment shown: $W \approx \vec{\mathbf{F}}(\vec{\mathbf{r}}) \bullet \Delta\vec{\mathbf{r}}$

and along the entire path C: $W \approx \sum_{i=1}^{n} \vec{\mathbf{F}}(\vec{\mathbf{r}}) \bullet \Delta\vec{\mathbf{r}}$

Taking the limit of a large number of subpaths: $$\int_C \vec{\mathbf{F}}(\vec{\mathbf{r}}) \bullet d\vec{r} = \lim_{n\to\infty}\left(\sum_{i=1}^{n} \vec{\mathbf{F}}(\vec{\mathbf{r}}) \bullet \Delta\vec{\mathbf{r}}\right)$$

$\int_C \vec{\mathbf{F}}(\vec{\mathbf{r}}) \bullet d\vec{\mathbf{r}}$ is the **scalar line integral** of $\vec{\mathbf{F}}(\vec{\mathbf{r}})$ along the path C.

Work continued ...

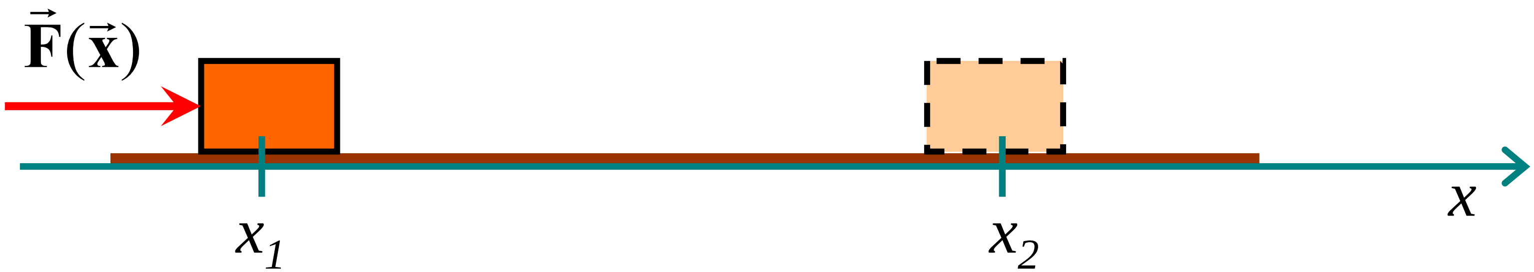
2 special cases:

(i) If the force is not variable, i.e. $\vec{\mathbf{F}}$ is constant, and $\Delta\vec{\mathbf{r}}$ is in a straight line, then

$$W = \vec{\mathbf{F}} \bullet \Delta\vec{\mathbf{r}} = \left|\vec{\mathbf{F}}\right|\left|\Delta\vec{\mathbf{r}}\right|\cos\theta = F\Delta r\cos\theta$$

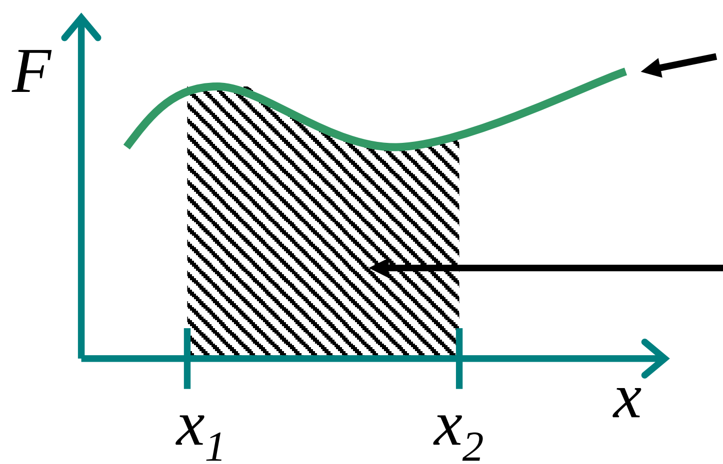
(ii) If $\vec{\mathbf{F}}$ is variable, but the displacement is still in only one direction,

then $W = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}) \bullet d\vec{\mathbf{r}}$ simplifies to $W = \int_{x_1}^{x_2} \vec{\mathbf{F}}(\vec{\mathbf{x}}) \bullet d\vec{\mathbf{x}}$



This integral may be interpreted as the area under the F - x curve.

For example:

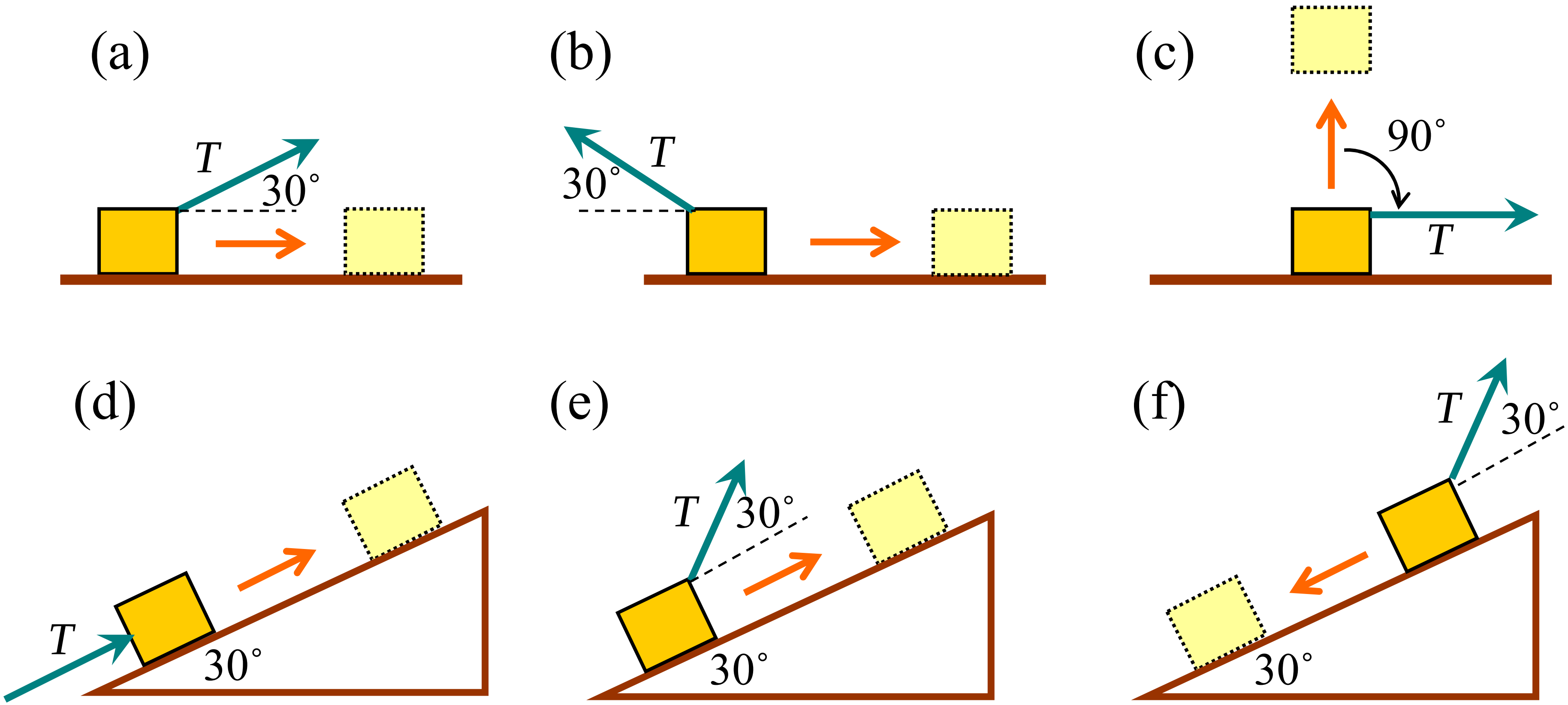


Applied force which varies as a function of position x.

Work done by F in moving the object from x_1 to x_2.

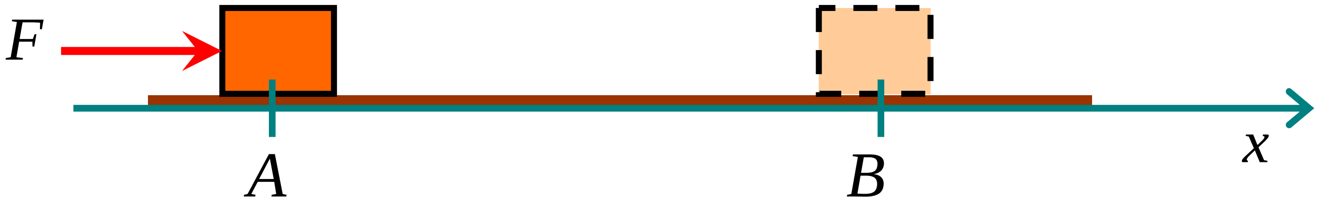
The work done by a force.

A force $T = 50$ N acts on a block as it moves a distance of 30 m in a straight line as shown in each case below. Calculate the work done by the force T in each case.

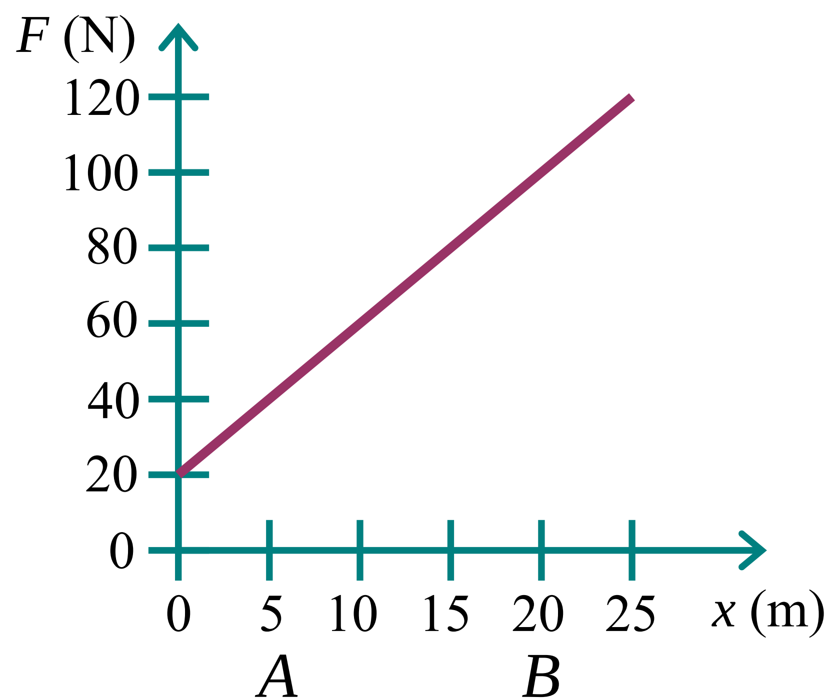


Work done by a force

A force F moves a box from point A to point B along the x-axis.



Shown alongside is the force-distance graph for this situation.



(a) From the graph, calculate the work done by this force in moving the box from A to B.

(b) Using an appropriate integral, also calculate this work.

Example

A force $\vec{\mathbf{F}} = 3\hat{\mathbf{i}} - 5\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ N acts on a 10 kg mass and moves it from an initial position $\vec{\mathbf{r}}_i = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$ m to a final position $\vec{\mathbf{r}}_f = 5\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$ m. How much work is done by this force in moving the block through this displacement?

Energy

If the work done on a closed system is **positive**, then energy is transferred to the system.
If the work done on a closed system is **negative**, then energy is removed from the system.

... but what is energy?

Types of energy

1. Kinetic energy, *K*

... the energy possessed by a system by virtue of its motion.

$$K = \tfrac{1}{2} m v^2$$

$K = 0$, if the body's speed $v = 0$.

Unit of energy: joule (J)

Energy is a **scalar** quantity.

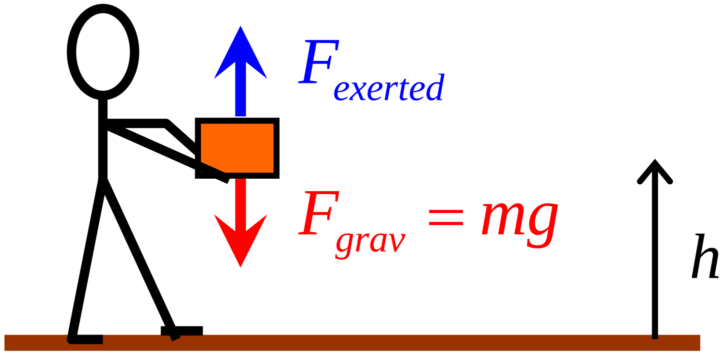
2. Potential energy, *U*

... the energy possessed by a system by virtue of its position or shape.

There are many different types of potential energy. We will focus on gravitational potential energy and "spring" potential energy.

2(a). Gravitational potential energy U_g .

Consider a mass *m* being lifted at a constant speed a height *h* by an external agent in the gravitational field of the earth.



$$\vec{\mathbf{F}}_{exerted} = -\vec{\mathbf{F}}_{grav}$$

$$\therefore F_{exerted} = mg$$

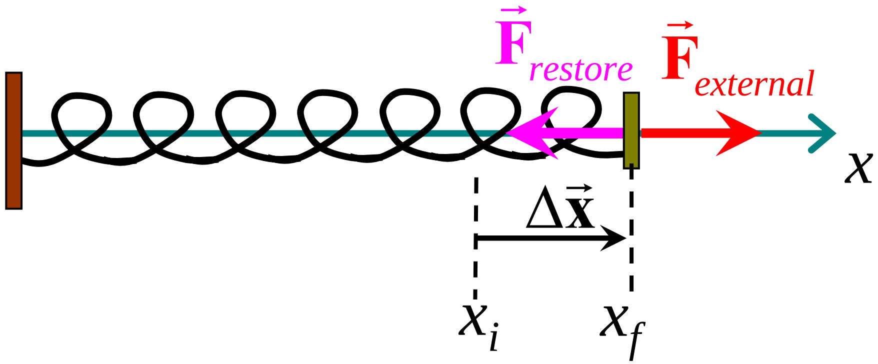
Work done by $\vec{\mathbf{F}}_{exerted}$: $W_{person} = \vec{\mathbf{F}}_{exerted} \bullet \Delta\vec{\mathbf{r}} = F_{exerted} h \cos 0° = mgh$

Work done by $\vec{\mathbf{F}}_{grav}$: $W_{grav} = \vec{\mathbf{F}}_{grav} \bullet \Delta\vec{\mathbf{r}} = F_{grav} h \cos 180° = -mgh$

The work done by $\vec{\mathbf{F}}_{exerted}$ on the box is *mgh*. In other words, at height *h*, the box has the ability to do an amount of work *mgh*. When the box is released at height *h*, a positive amount of work is done by $\vec{\mathbf{F}}_{grav}$ and the box accelerates and gains kinetic energy.

2(b). The potential energy associated with a compressed or extended spring U_s .

$$\vec{\mathbf{F}}_{restore} = -\vec{\mathbf{F}}_{external}$$



Hooke's Law:

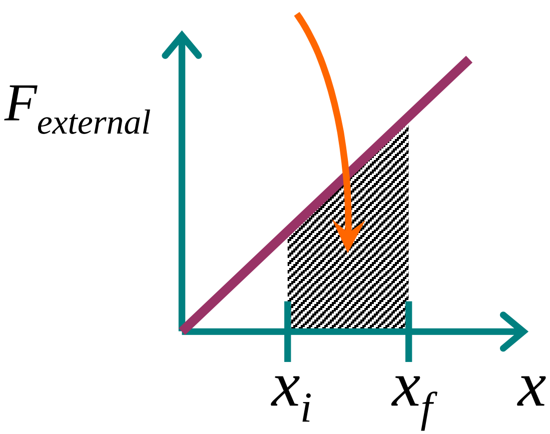
$$\vec{\mathbf{F}}_{restore} = -k\Delta\vec{\mathbf{x}}$$

where k is the "spring constant" [N m^{-1}]

The work done by $\vec{\mathbf{F}}_{external}$ to extend the spring from x_i to x_f :

$$W_{F_{external}} = \int_{x_i}^{x_f} \vec{\mathbf{F}}_{external} \bullet d\vec{\mathbf{x}} = \int_{x_i}^{x_f} F dx = \int_{x_i}^{x_f} kx dx = \tfrac{1}{2}kx^2\Big|_{x_i}^{x_f}$$

$$= \tfrac{1}{2}kx_f^2 - \tfrac{1}{2}kx_i^2$$



Therefore the potential energy stored in the spring ΔU_s :

$$\Delta U_s = \tfrac{1}{2}kx_f^2 - \tfrac{1}{2}kx_i^2$$

3. Energy lost to dissipative processes

A bullet is fired horizontally into a wall. Before it strikes the wall, it is travelling at 500 m s^{-1}. After it comes to rest in the wall, its speed is zero. Where did all the kinetic energy go to?

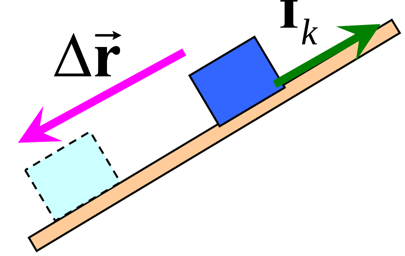
Let's look at the impact itself. Firstly, the sound of the impact carries away some energy. Some energy is required to distort the wall (and the bullet). However, most energy is lost as **heat**. The impact causes vibrations of the molecules of the wall and bullet and both are a little warmer after the impact.

3. Energy lost to dissipative processes ... continued

Even when all attempts are made to prevent dissipative energy losses to the environment, there will always be some loss. Think about all those attempts to build "perpetual motion" machines. There is always some friction in mechanical systems.

In this course, we will only consider the energy lost due to sliding friction.

For a block sliding on a rough surface:



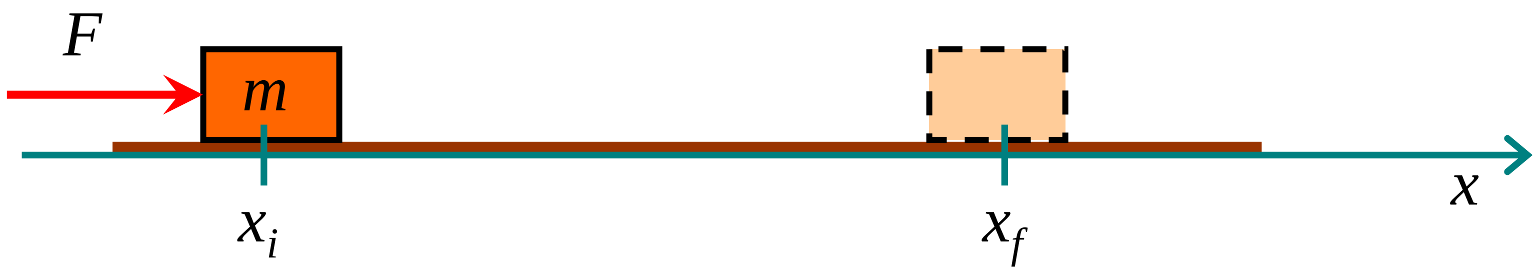
The work done by the friction force W_{f_k} = the energy lost to friction $\Delta U_{friction}$

$$W_{f_k} = \Delta U_{friction} = \vec{\mathbf{f}}_k \bullet \Delta\vec{\mathbf{r}} = -f_k \Delta r$$

Note the minus sign

The work-energy theorem

Consider a mass m moving along the x-axis acted upon by a constant force F:



Then the work done by F in moving m from x_i to x_f :

$$W = \int_{x_i}^{x_f} F dx = \int_{x_i}^{x_f} madx$$

But $\quad madx = m\frac{dv}{dt}dx = m\frac{dx}{dt}dv = mvdv$

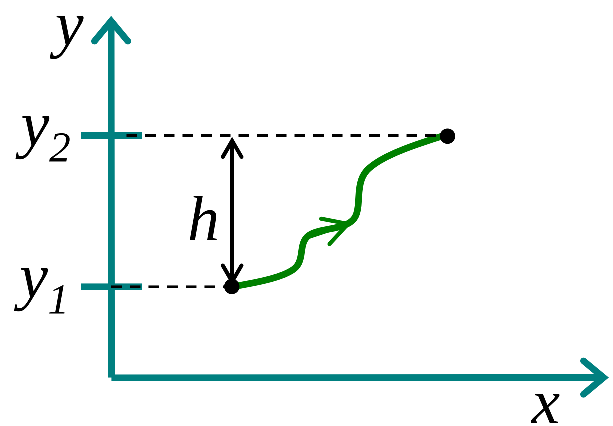
$$\therefore W = \int_{v_i}^{v_f} mvdv = m\int_{v_i}^{v_f} mvdv = \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2$$

$$\therefore W = K_f - K_i = \Delta K$$

If W is positive, then the mass will speed up.
If W is negative, then the mass will slow down.

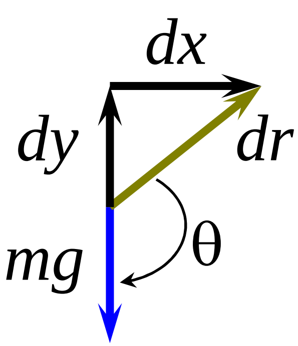
Conservative and non-conservative forces

Consider a mass m being lifted by an external agent in the gravitational field of the earth from vertical positions y_1 to y_2 along a curved path as shown alongside.



The work done W_{grav} by the gravitational force on the mass:

$$W_{grav} = \int_1^2 \vec{\mathbf{F}}_{grav} \bullet d\vec{\mathbf{r}} = \int_1^2 mg\cos\theta dr$$



$$\cos(180° - \theta) = \frac{dy}{dr}$$

$$\therefore dr(-\cos\theta) = dy$$

$$W_{grav} = -\int_{y_1}^{y_2} mgdy$$

$$\therefore W_{grav} = -mg(y_2 - y_1) = -mgh$$

W_{grav} is path independent (depends on $y_2 - y_1$ only).

Conservative and non-conservative forces ... 2

Since W_{grav} is **path independent**, we call $\vec{\mathbf{F}}_{grav}$ a **conservative force**.

... and the change in gravitational potential energy ΔU_g when a mass m moves from height y_1 to y_2 can be defined as

$$\Delta U_g = U_2 - U_1 = -W_{grav} = -\int_1^2 \vec{\mathbf{F}}_{grav} \bullet d\vec{\mathbf{r}} = +mg(y_2 - y_1)$$

... can write $\Delta U_g = U(r) - U(r_0) = -\int_1^2 \vec{\mathbf{F}}_{grav} \bullet d\vec{\mathbf{r}}$

$$\therefore U(r) = -\int_1^2 \vec{\mathbf{F}}_{grav} \bullet d\vec{\mathbf{r}} + U(r_0)$$

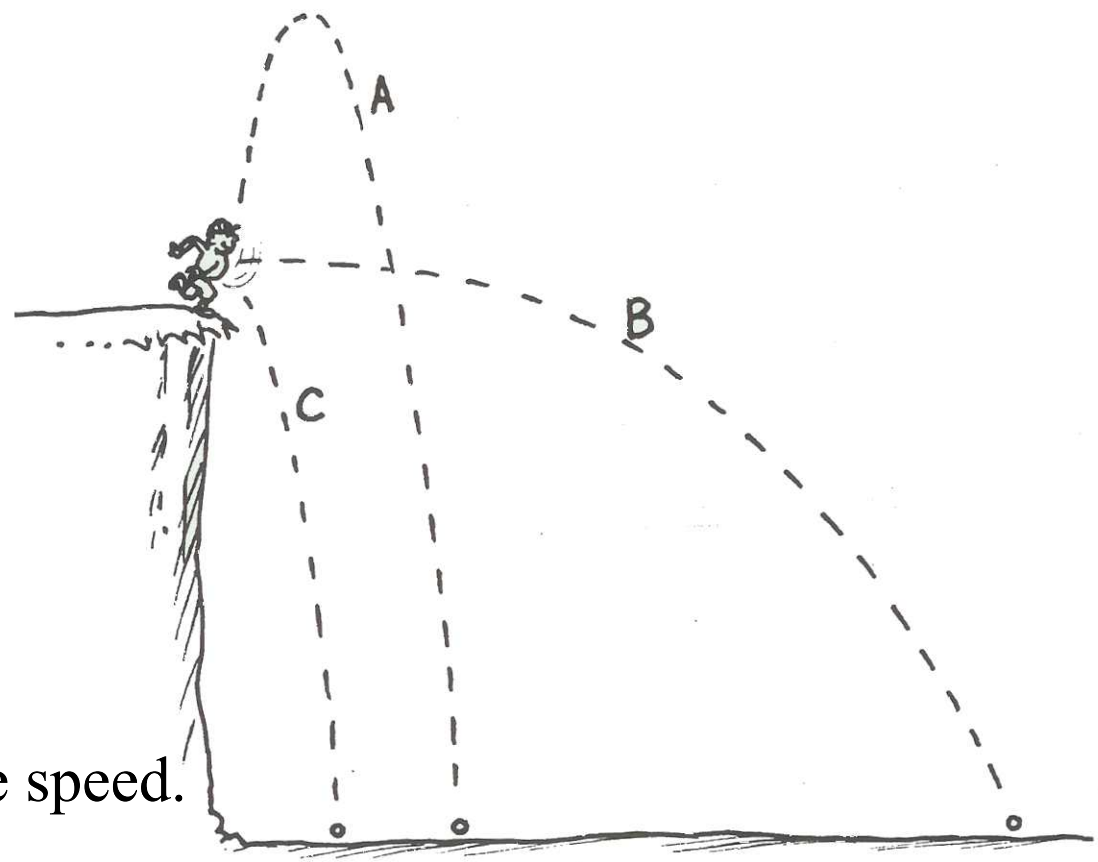
... can choose $U(r_0) = 0$ arbitrarily.

Therefore only ΔU_g is physically meaningful.

Not all forces are conservative.
For example, friction forces are always **non-conservative**.

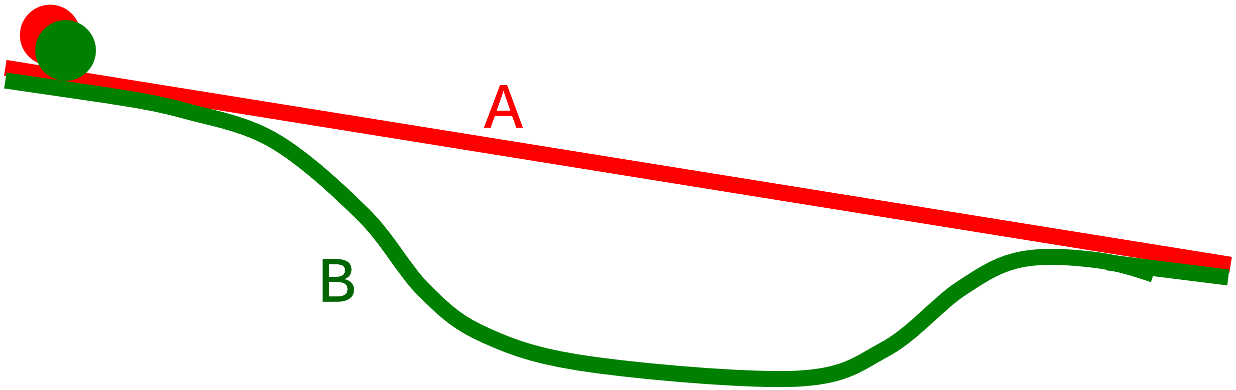
Figuring Physics

Three balls are thrown from the top of a cliff with the same initial speed along paths A, B and C. If there is no air resistance, which ball will strike the ground with the greatest speed?



(A) A
(B) C
(C) all strike with the same speed.

Demonstration

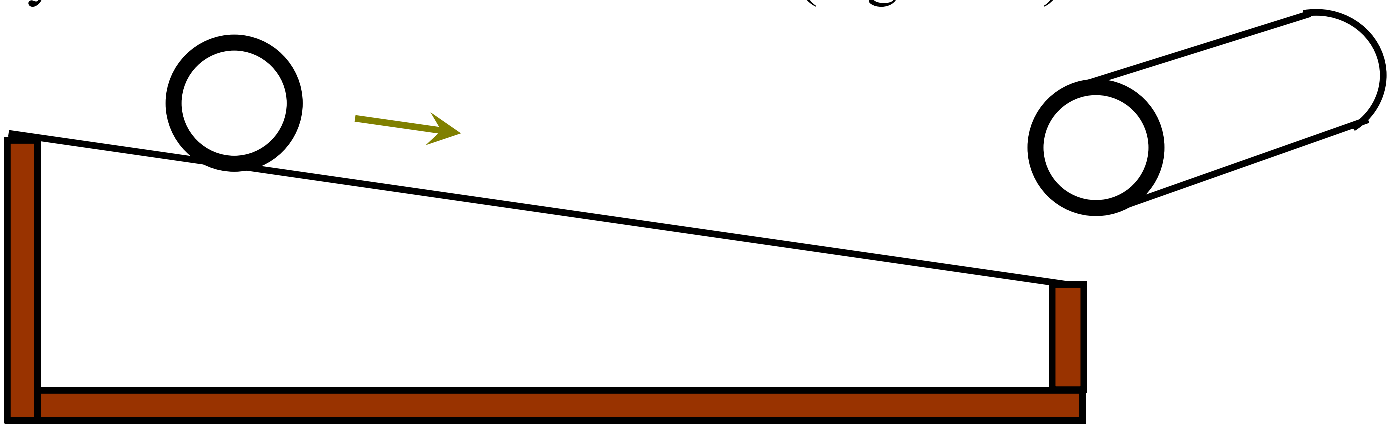


Two identical balls are able to roll long two smooth tracks. Track A is a straight downwards slope. Track B has a curved path as shown. Both tracks start and end at the same vertical position. If the two balls are released from rest at the top of the tracks, which ball will reach the bottom first?

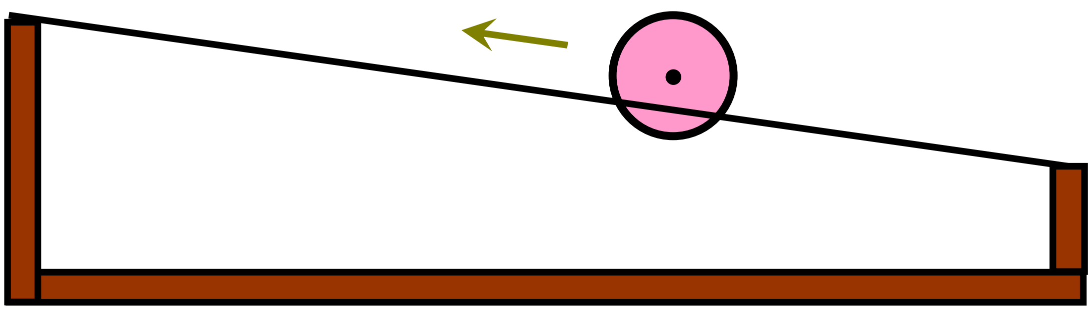
(a) A
(b) B
(c) both A and B at the same time.

Demonstration

A cylinder rolls down the incline (big deal!)



But the other object appears to be rolling uphill!



What is going on?

Principle of Conservation of Energy

For an isolated, closed system energy can be transformed from one form to another, but the total energy remains constant.
If an external force acts on the system, then energy may either be added to the system or taken away.

For a particular isolated system:

$$\Delta K + \Delta U_g + \Delta U_s + \Delta(\text{other forms of energy}) = 0$$

If external forces and friction act on the system:

$$\Delta K + \Delta U_g + \Delta U_s + \Delta(\text{other forms of energy}) = W_{ext} - \Delta U_{friction}$$

Work done by an external force

Energy lost to friction

Conservation of Energy

For an isolated system:

$$\Delta K \quad + \quad \Delta U_g \quad + \quad \Delta U_s \quad = \quad W_{ext} - \Delta U_{friction}$$

or $$\left(K^f - K^i\right) + \left(U_g^f - U_g^i\right) + \left(U_s^f - U_s^i\right) = W - \Delta U_{friction}$$

initial situation final situation

$$K^i + U_g^i + U_s^i + W_{ext} = K^f + U_g^f + U_s^f + \Delta U_{friction}$$

$$\tfrac{1}{2}mv_i^2 + mgy_i + \tfrac{1}{2}kx_i^2 + \sum \vec{\mathbf{F}}_{external} \bullet \Delta\vec{\mathbf{r}} = \tfrac{1}{2}mv_f^2 + mgy_f + \tfrac{1}{2}kx_f^2 + \vec{\mathbf{f}}_k \bullet \Delta\vec{\mathbf{r}}$$

Conservation of Energy

Solving physics problems by using the principle of conservation of energy is essentially an energy "book-keeping" exercise.

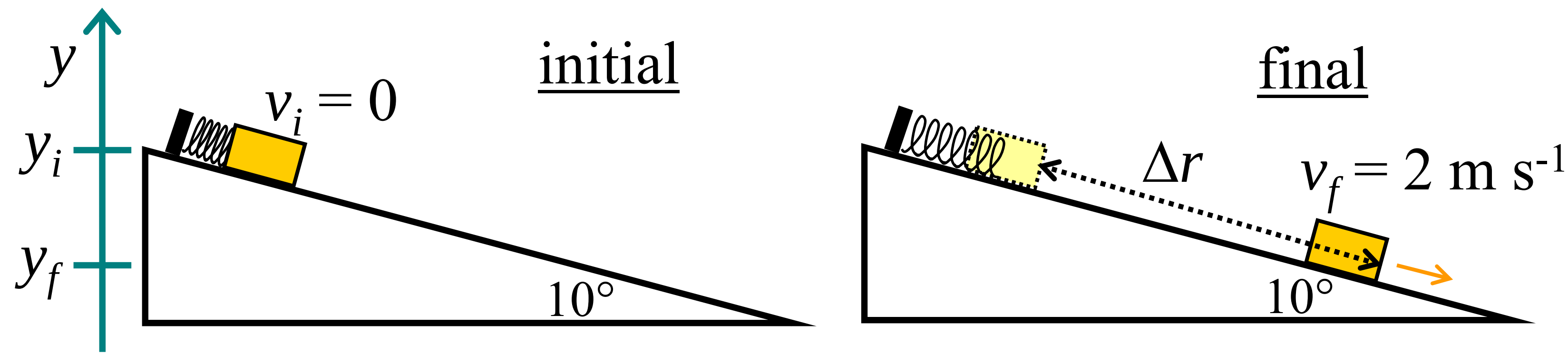
1. Identify your system carefully.
2. Identify the initial and final situations.
3. Identify which of $K^i, K^f, U_g^i, U_g^f, U_s^i, U_s^f$ are present. Note that usually either y_i or y_f will be set as zero.
4. Is there a frictional force acting? (Is $\Delta U_{friction}$ zero?)
5. Are there other external forces acting on the system? (Is W_{ext} zero?)
6. Apply the master energy equation to the problem:

$$K^i + U_g^i + U_s^i + W_{ext} = K^f + U_g^f + U_s^f + \Delta U_{friction}$$

7. Simply this equation down to the required components.
8. Solve for what is required.

Example 1

A 15 kg block is held against a spring having $k = 8000\ \mathrm{N\ m^{-1}}$, compressing it by 20 cm. The block is released to slide down a rough slope inclined at 10˚. What distance down the slope is the block moving at $2\ \mathrm{m\ s^{-1}}$? The coefficient of kinetic friction between the block and the slope is 0.6 .



$$K^i + U_g^i + U_s^i + W_{ext} = K^f + U_g^f + U_s^f + \Delta U_{friction}$$

K^i	U_g^i	U_s^i	W_{ext}	K^f	U_g^f	U_s^f	$\Delta U_{friction}$
✗	✓	✓	✗	✓	✗	✗	✓

↑ set $y_f = 0$ (for U_g^f)

$$K^i \; + \; U_g^i \; + \; U_s^i \; + \; W_{ext} = K^f \; + \; U_g^f \; + \; U_s^f + \Delta U_{friction}$$

✗ ✓ ✓ ✗ ✓ ✗ ✗ ✓

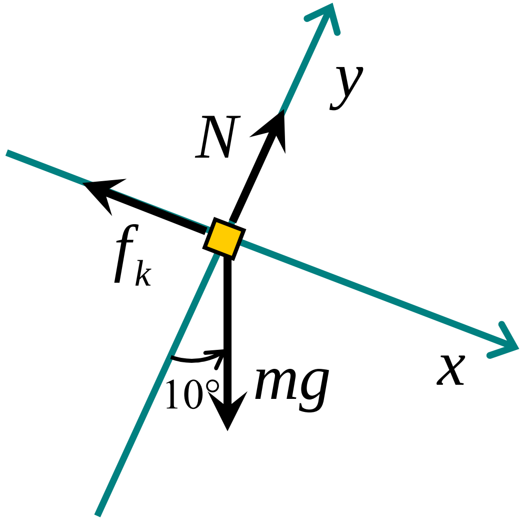
$$mgy_i + \tfrac{1}{2}kx_i^2 = \tfrac{1}{2}mv_f^2 + \vec{\mathbf{f}}_k \bullet \Delta\vec{\mathbf{r}}$$

$$mgy_i + \tfrac{1}{2}kx_i^2 = \tfrac{1}{2}mv_f^2 + \mu_k N \Delta r$$

$$mg\left(\Delta r \sin 10^\circ\right) + \tfrac{1}{2}kx_i^2 = \tfrac{1}{2}mv_f^2 + \mu_k\left(mg\cos 10^\circ\right)\Delta r$$

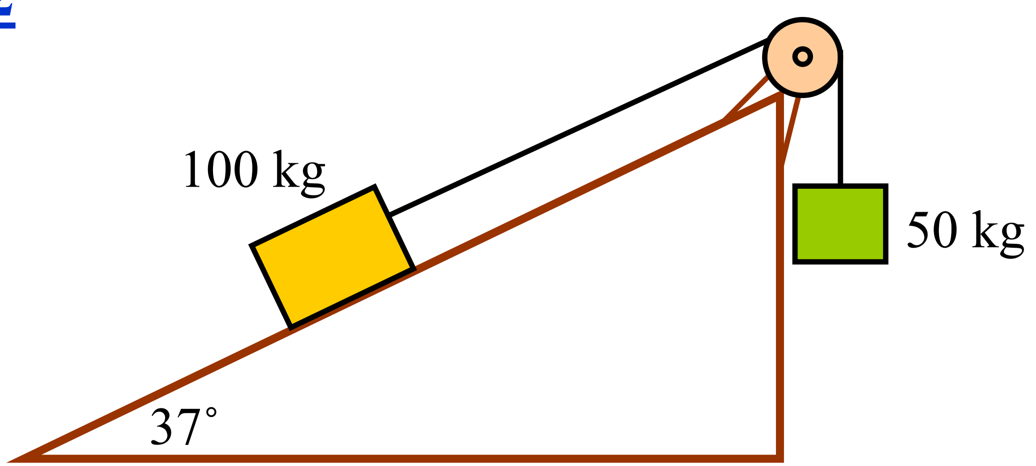
$$\therefore (15)(9.8)\Delta r\left(\sin 10^\circ\right) + \tfrac{1}{2}(8000)(0.2)^2$$

$$= \tfrac{1}{2}(15)(2)^2 + (0.6)(15)(9.8)(\cos 10^\circ)\Delta r$$



$$\therefore \Delta r = 1.69 \text{ m}$$

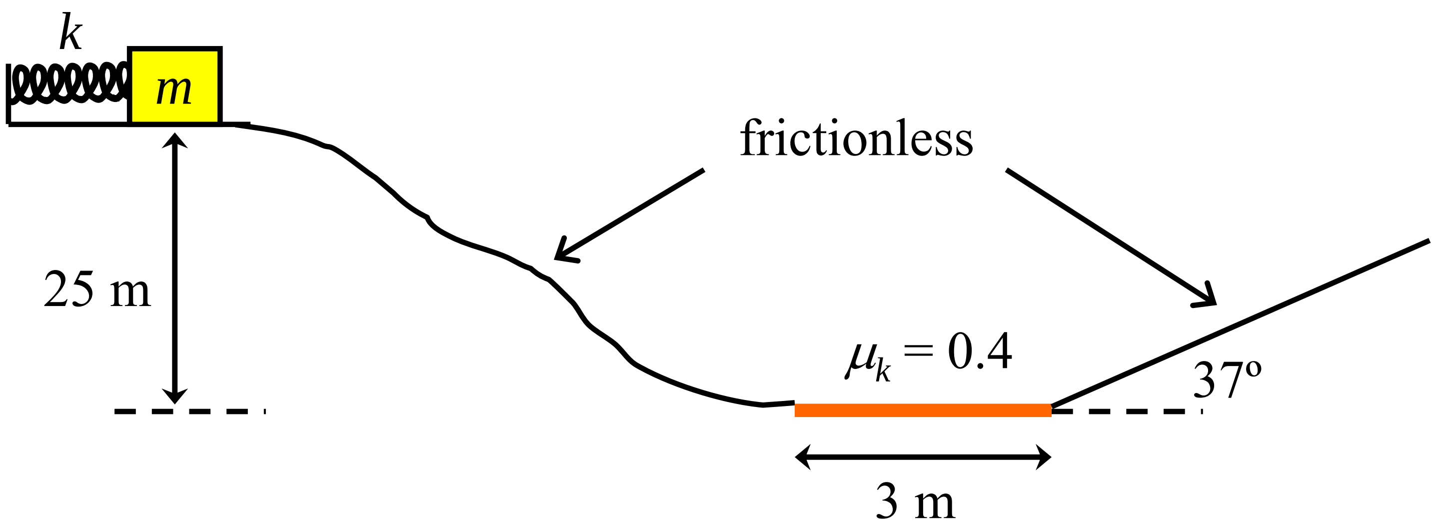
Example 2



A 100 kg block, initially at rest, slides down a 37º frictionless incline. A rope attached to the back of the block passes over a massless, frictionless pulley and down to a 50 kg hanging mass. Determine the speed of the two blocks after moving 10 m.

Example 3

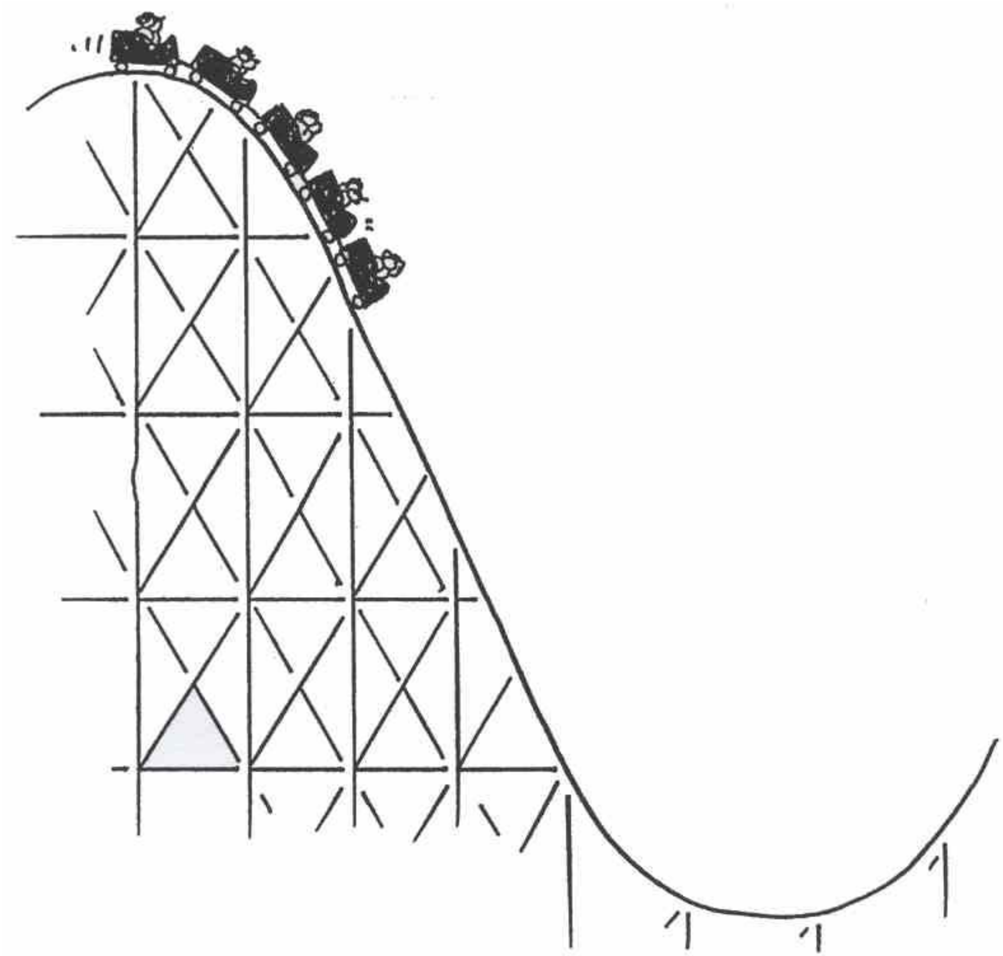
A block of mass 5 kg is held against a spring with a spring constant of 2000 N m^{-1} and which is compressed by 20 cm. The block is then released and is pushed to slide down a frictionless hill and across a rough surface having $\mu_k = 0.4$ and then up a frictionless slope included at 37º to the horizontal. How far along this slope does the block come to rest?



Figuring Physics

In which car will you be moving fastest at the very bottom of the incline?

(A) front car
(B) middle car
(C) rear car



Power

Power P is the rate at which work W is done.

Unit of power: the watt, W (1 watt = 1 joule second^{-1})

Average power: $$P_{av} = \frac{W}{\Delta t} = \frac{\Delta U}{\Delta t}$$

Instantaneous power: $$P = \frac{dW}{dt}$$

... can also write: $$P = \frac{dW}{dt} = \frac{d}{dt}\left(\vec{\mathbf{F}}\bullet\vec{\mathbf{r}}\right) = \vec{\mathbf{F}}\bullet\frac{d\vec{\mathbf{r}}}{dt} = \vec{\mathbf{F}}\bullet\vec{\mathbf{v}}$$

Linear momentum

Linear momentum for a single particle: $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$

Recall that Newton II can be expressed as $\vec{\mathbf{F}}_{ext} = \dfrac{d\vec{\mathbf{p}}}{dt}$

For a system of particles:

$$\vec{\mathbf{p}} = \sum_i \vec{\mathbf{p}}_i = m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2 + \ldots + m_n\vec{\mathbf{v}}_n = \sum_i m_i\vec{\mathbf{v}}_i = M\vec{\mathbf{v}}_{cm}$$

Newton II for a system of particles:

$$\frac{d\vec{\mathbf{p}}}{dt} = M\frac{d\vec{\mathbf{v}}_{cm}}{dt} = M\vec{\mathbf{a}}_{cm} = \vec{\mathbf{F}}_{ext}$$

Principle of the conservation of linear momentum

$$\vec{\mathbf{F}}_{ext} = \frac{d\vec{\mathbf{p}}}{dt}$$

If the external force on a system of particles is zero $\vec{\mathbf{F}}_{ext} = 0$, then

$$\frac{d\vec{\mathbf{p}}}{dt} = 0$$

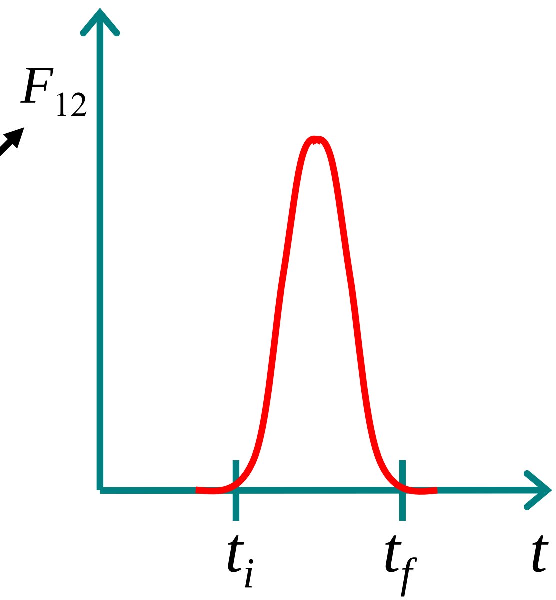
or $\vec{\mathbf{p}} = \text{constant}$

If there is no resultant external force acting on an isolated system, then the total linear momentum remains constant.

Impulse and momentum

A collision is when two bodies interact over a short time interval. The forces that the bodies exert on each other are usually so strong during the collision that all forces acting on a body may be ignored.

During a collision between two bodies (1 and 2), the contact force exerted by one body on the other jumps from zero to a very large value and then abruptly drops to zero again.



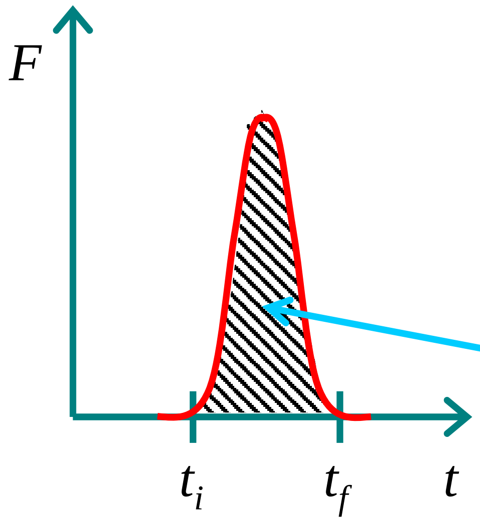
The time interval $\Delta t = t_f - t_i$ is usually very small.

Note that $\vec{\mathbf{F}}_{12} = -\vec{\mathbf{F}}_{21}$ for the collision

Impulse and momentum ... 2

$$\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt} \quad \text{for each object} \quad \rightarrow \vec{\mathbf{F}}dt = d\vec{\mathbf{p}}$$

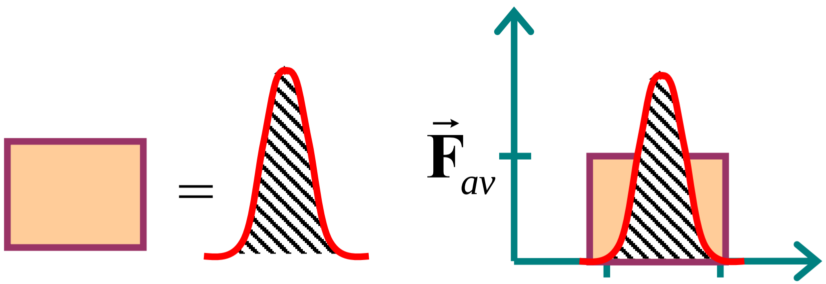
Integrating over the time of the collision:



$$\underbrace{\int_{t_i}^{t_f} \vec{\mathbf{F}}dt}_{} = \int_{p_i}^{p_f} d\vec{\mathbf{p}}$$

Impulse $\vec{\mathbf{J}} = \vec{\mathbf{p}}_f - \vec{\mathbf{p}}_i = \Delta\vec{\mathbf{p}}$

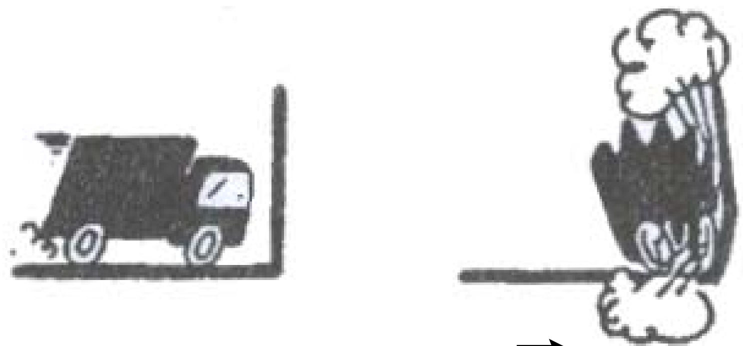
Sometimes it is useful to use the average force $\vec{\mathbf{F}}_{av}$ acting for time Δt to give the same impulse $\vec{\mathbf{J}}$ and $\Delta\vec{\mathbf{p}}$.



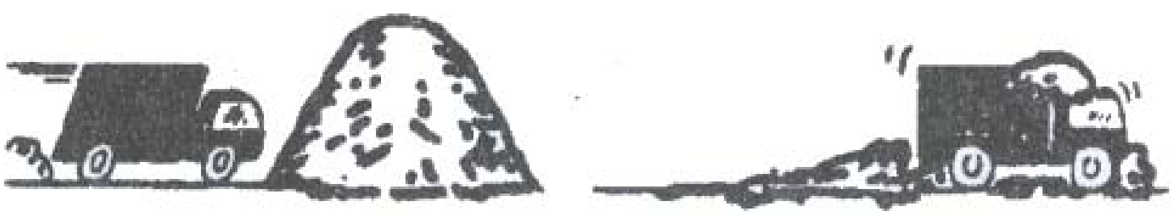
Then $\vec{\mathbf{F}}_{av}\Delta t = \vec{\mathbf{J}} = \Delta\vec{\mathbf{p}}$

Impulse and momentum ... 3

The impulse $\vec{\mathbf{J}}$ is the same in each case.



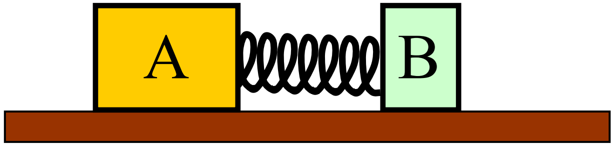
$$m\,\Delta\vec{\mathbf{v}} = \vec{\mathbf{F}}\,\Delta t$$



$$m\,\Delta\vec{\mathbf{v}} = \vec{\mathbf{F}}\,\Delta t$$

Thinking about momentum 1

Two blocks A and B, rest on a horizontal frictionless table. The blocks are separated by a compressed spring of negligible mass. The mass of block A is twice that of block B. When the blocks are released, they move apart. Which one of the following statements is true?



I.

(a) The magnitude of the momentum of A equals that of B after release.
(b) The magnitude of the momentum of A is greater than B after release.
(c) The magnitude of the momentum of A is less than B after release.

II.

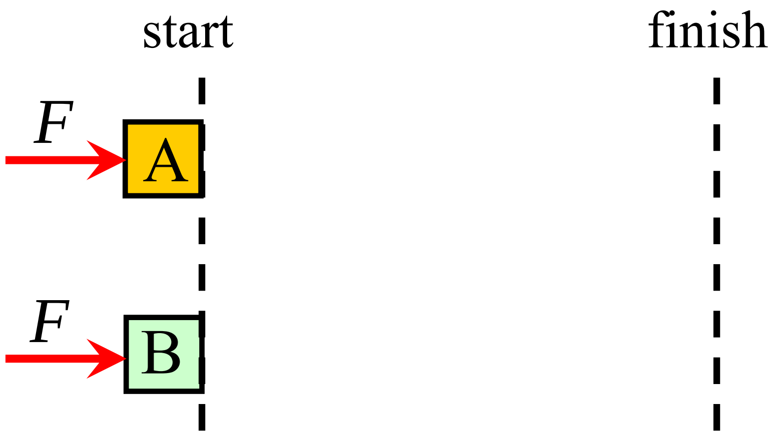
(a) The total momentum of the blocks after release is the same as before release.
(b) The total momentum of the blocks after release is greater than before release.
(c) The total momentum of the blocks after release is less than before release.

III.

(a) The speed of block A is the same as that of B at all times after release.
(b) The speed of block A is greater than that of B after release.
(c) The speed of block A is less than that of B after release.

Thinking about momentum 2

Identical constant forces continuously push identical blocks A and B from the start line to the finish line.
Block A is initially at rest.
Block B is initially moving to the right.



I. Which block has the larger change in momentum?

(a) A
(b) B
(c) They have the same momenta change.

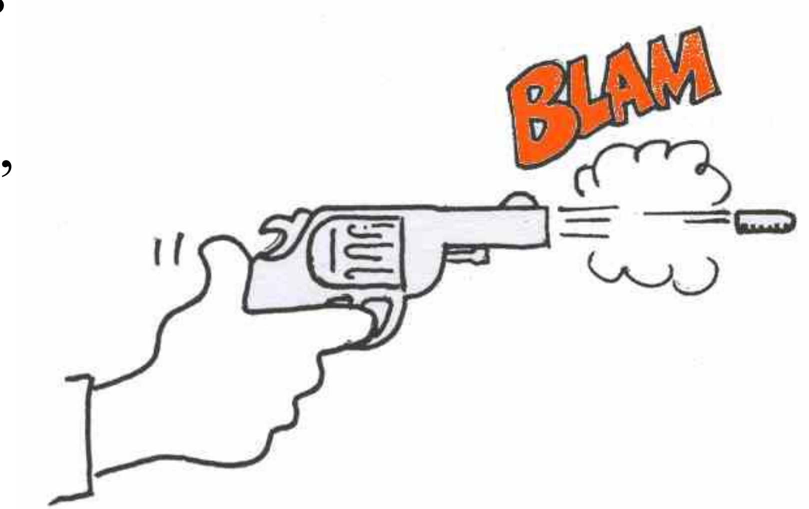
II. My reason for my answer to I is

(a) The same force acts on identical blocks for the same distance.
(b) Block B already has some momentum, so its change isn't as great.
(c) The impulse on block B is less since the force acts for a shorter time interval.
(d) Block B is moving faster at the finish line, so its change is greater.
(e) The initial and final velocities are not given.

Figuring Physics

Strictly speaking, when a gun is fired, compared with the momentum of the recoiling gun, the opposite momentum of the bullet is

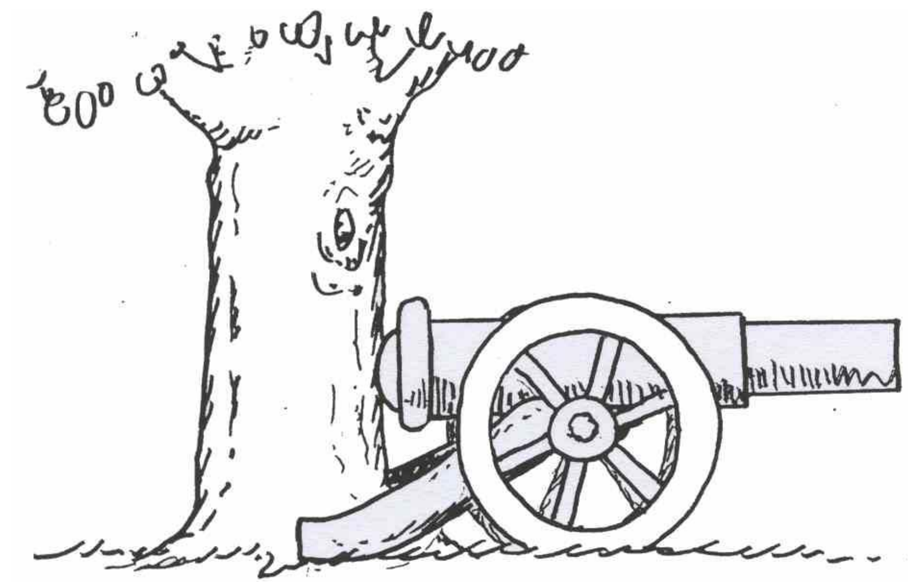
(A) less
(B) more
(C) the same



FIGURING PHYSICS

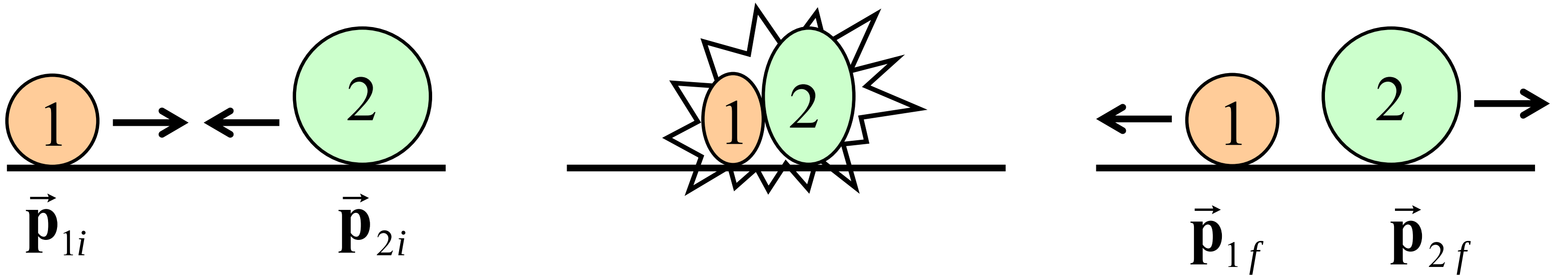
Suppose a cannon is propped against a massive tree to reduce recoil when it fires, Then the range of the cannonball will be

(A) increased
(B) decreased
(C) unchanged



More on collisions

Consider the collision of two bodies:



$$\Delta\vec{\mathbf{p}}_1 = \vec{\mathbf{p}}_{1f} - \vec{\mathbf{p}}_{1i} = \int_{t_i}^{t_f} \vec{\mathbf{F}}_{21}\,dt$$

and $$\Delta\vec{\mathbf{p}}_2 = \vec{\mathbf{p}}_{2f} - \vec{\mathbf{p}}_{2i} = \int_{t_i}^{t_f} \vec{\mathbf{F}}_{12}\,dt$$

But $$\vec{\mathbf{F}}_{21} = -\vec{\mathbf{F}}_{12}$$

$$\therefore \Delta\vec{\mathbf{p}}_1 = -\Delta\vec{\mathbf{p}}_2$$

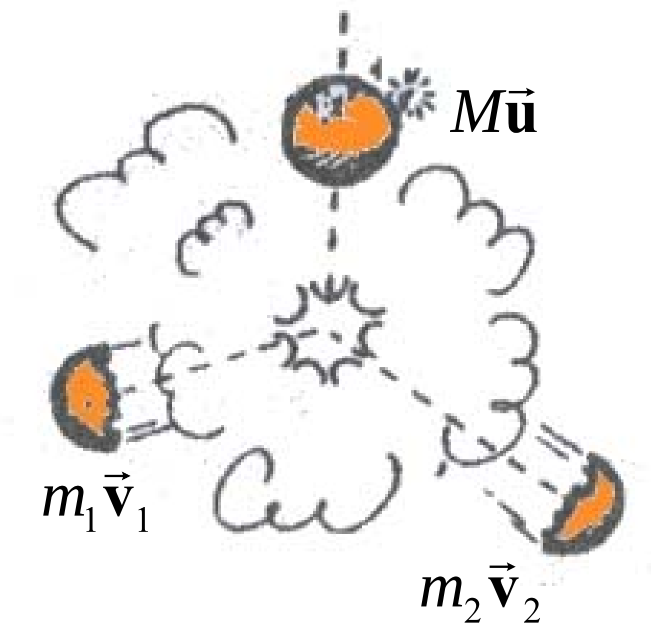
$$\therefore \vec{\mathbf{p}}_{1f} - \vec{\mathbf{p}}_{1i} = -(\vec{\mathbf{p}}_{2f} - \vec{\mathbf{p}}_{2i})$$

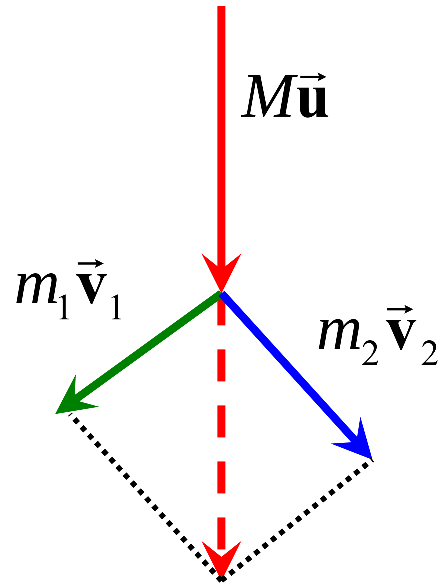
$$\therefore \vec{\mathbf{p}}_{1i} + \vec{\mathbf{p}}_{2i} = \vec{\mathbf{p}}_{1f} + \vec{\mathbf{p}}_{2f}$$

Momentum is conserved.

or $$m_1\vec{\mathbf{u}}_1 + m_2\vec{\mathbf{u}}_2 = m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2$$

Another case ... an exploding object





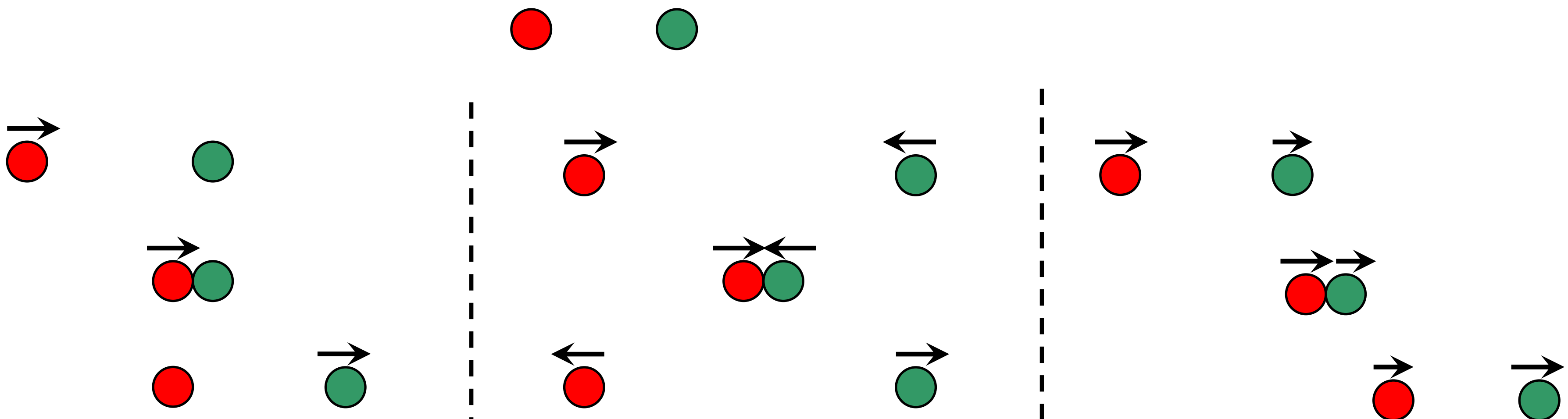
$$M\vec{\mathbf{u}} = m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2$$

Elastic collisions ...

both linear momentum and kinetic energy are conserved.

$$m_1\vec{\mathbf{u}}_1 + m_2\vec{\mathbf{u}}_2 = m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2 \quad \text{and} \quad \tfrac{1}{2}m_1u_1^2 + \tfrac{1}{2}m_2u_2^2 = \tfrac{1}{2}m_1v_2^2 + \tfrac{1}{2}m_2v_2^2$$

Three examples for $m_1 = m_2$:



$\vec{\mathbf{u}}_1 = \vec{\mathbf{u}}_1$

$\vec{\mathbf{u}}_2 = 0$

$\vec{\mathbf{v}}_1 = 0$

$\vec{\mathbf{v}}_2 = \vec{\mathbf{u}}_1$

$\vec{\mathbf{u}}_1 = \vec{\mathbf{u}}_1$

$\vec{\mathbf{u}}_2 = -\vec{\mathbf{u}}_1$

$\vec{\mathbf{v}}_1 = -\vec{\mathbf{u}}_1$

$\vec{\mathbf{v}}_2 = \vec{\mathbf{u}}_1$

$\vec{\mathbf{u}}_1 = \vec{\mathbf{u}}_1$

$0 < \vec{\mathbf{u}}_2 < \vec{\mathbf{u}}_1$

$\vec{\mathbf{v}}_1 = \vec{\mathbf{u}}_2$

$\vec{\mathbf{v}}_2 = \vec{\mathbf{u}}_1$

Demonstration: “Newton’s cradle”

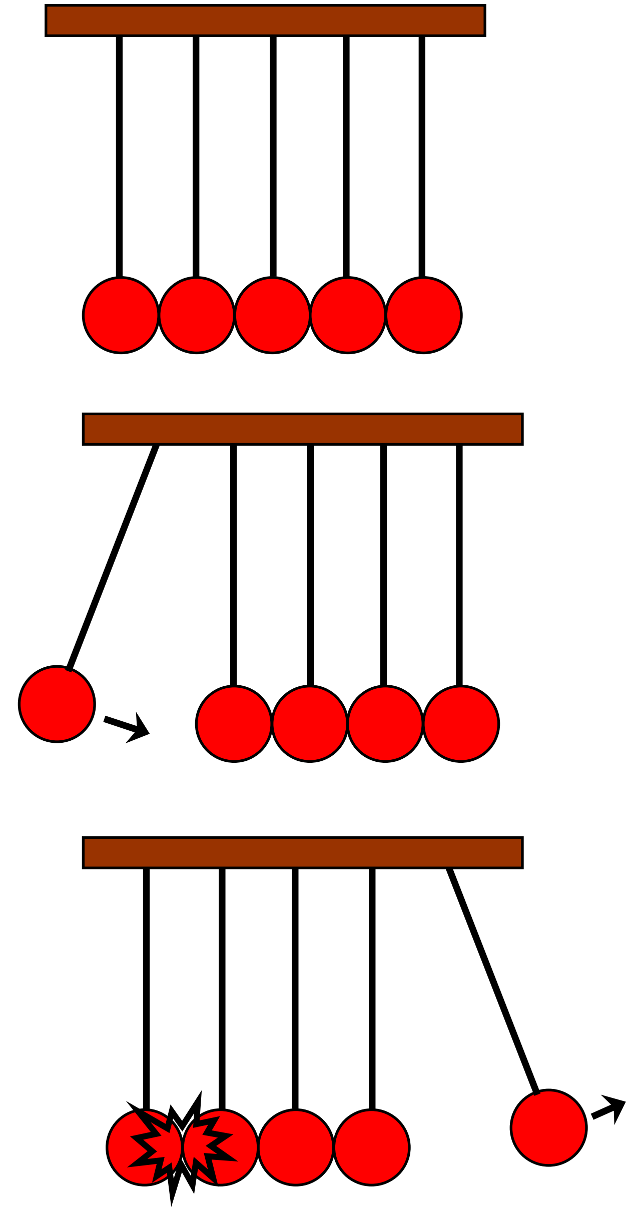
Five spheres of equal mass hang at the end of strings.

One ball is pulled back and released to strike the other stationary balls...

... and one ball flies off on the other side.

Kinetic energy is conserved so the collision is elastic.

But why don’t two balls fly out with half the speed?



If two balls fly off with half the speed of the incoming ball, then that would conserve momentum, since

$$mv = \tfrac{1}{2}mv + \tfrac{1}{2}mv$$

But it wouldn't conserve kinetic energy.

The incoming ball has kinetic energy $\tfrac{1}{2}mv^2$

and two outgoing balls with half the speed have kinetic energy:

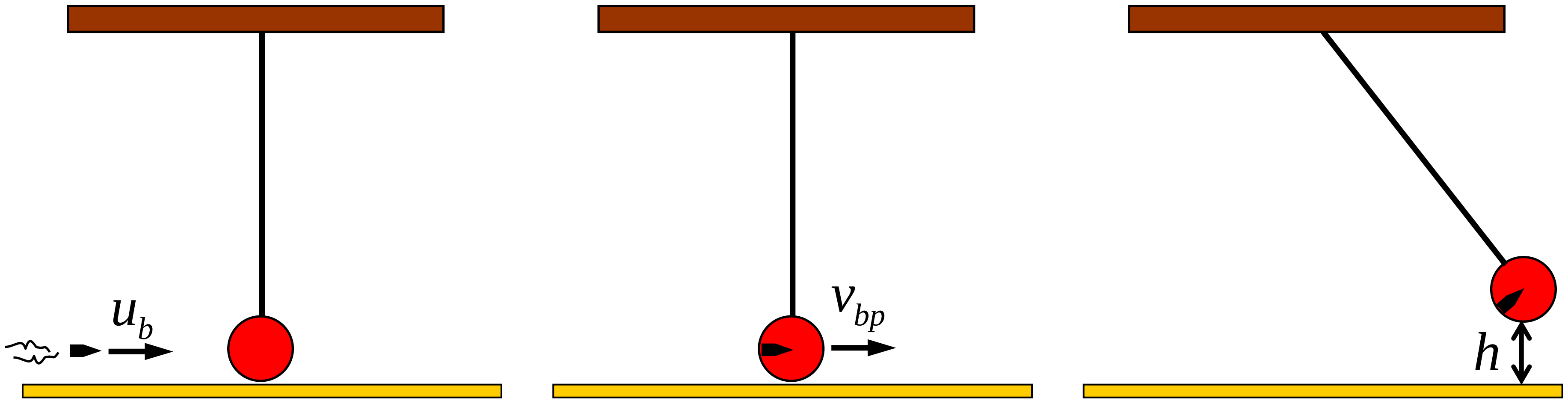
$$\tfrac{1}{2}m(\tfrac{1}{2}v)^2 + \tfrac{1}{2}m(\tfrac{1}{2}v)^2 = \tfrac{1}{4}mv^2 \neq \tfrac{1}{2}mv^2$$

On the other hand, in the case of **inelastic collisions,** momentum is conserved in the collision, but not kinetic energy.

"Perfectly inelastic collision" ... when the two bodies stick together.

Demonstration: The ballistic pendulum

1. A bullet is fired into a pendulum, which is initially at rest.
2. The bullet lodges in the pendulum, which moves to the right.
3. The bullet and pendulum swing to a height *h*.



$$m_b u_b + m_p u_p = (m_b + m_p) v_{bp}$$

$$m_b u_b + 0 = (m_b + m_p) v_{bp}$$

$$\tfrac{1}{2}(m_b + m_p) v_{bp}^2 = (m_b + m_p) g h$$

Figuring Physics

Whenever an interaction occurs in a system, forces occur in equal and opposite pairs. Which of the following do not always occur in equal and opposite pairs?

(A) Impulses
(B) Accelerations
(C) Momentum changes

Example 1

A 60 kg person standing on a 40 kg trolley are travelling to the right at a speed of 10 m s^{-1} on a horizontal surface. The person jumps off the trolley and flies through the air at a speed of 2 m s^{-1} to the left. Determine the velocity of the trolley relative to the ground immediately after the person jumps off. Assume that the axles of the trolley have a very low coefficient of friction.

Example 2

Ball A of mass 10 kg is moving at 30 m s^{-1} at 180 º.
Ball B of mass 20 kg is moving at 15 m s^{-1} at 300º.
They collide elastically and thereafter ball A is moving at 25 m s^{-1} at 45°.
All angles are measured anticlockwise from the positive *x*-axis.
What is the final velocity of ball B after the collision?

[Answer: $-16.3\hat{\mathbf{i}} - 21.8\,\hat{\mathbf{j}}$ m s^{-1}]

Rotational Dynamics

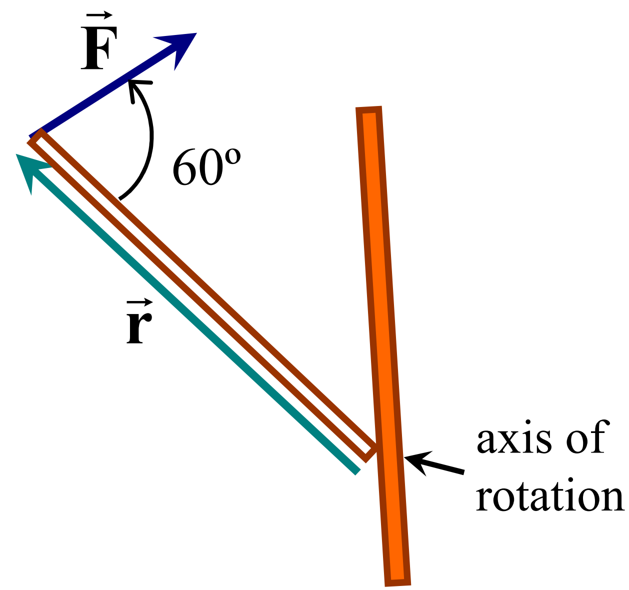
Torque

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$$

τ : "tau"

Also called the moment of the force $\vec{\mathbf{F}}$ about the turning point (axis of rotation).

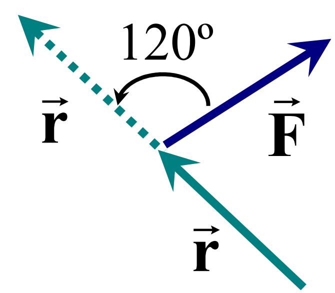
e.g. a metre stick is free to rotate about a fixed axis at one end as shown.



$$\tau = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$$

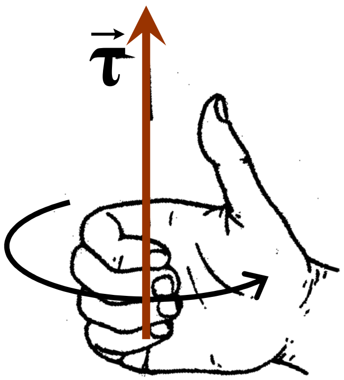
$$\left|\vec{\boldsymbol{\tau}}\right| = \tau = rF\sin\theta$$

θ is the angle between the tails of the $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$



$$\therefore \tau = (1)(4)\sin 120^\circ \;=\; 3.46 \text{ N m}$$

Direction of $\vec{\boldsymbol{\tau}}$:



Fingers of right hand in direction of rotation

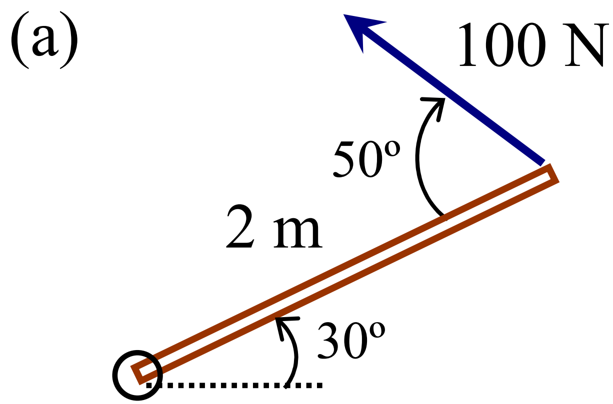
If $\vec{\mathbf{F}}$ causes a clockwise rotation then $\vec{\boldsymbol{\tau}}$ is into the page ($\vec{\boldsymbol{\tau}} = \tau \ (-\hat{\mathbf{k}})$)

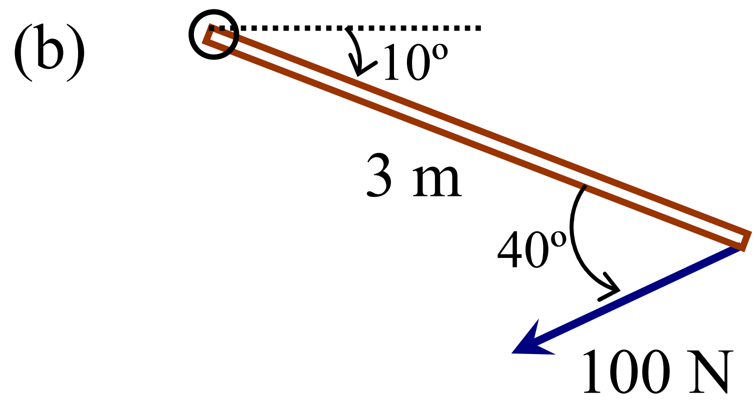
If $\vec{\mathbf{F}}$ causes a clockwise rotation then $\vec{\boldsymbol{\tau}}$ is into the page ($\vec{\boldsymbol{\tau}} = \tau \ (-\hat{\mathbf{k}})$)

In example above: $\vec{\boldsymbol{\tau}} = 3.46 \ (-\hat{\mathbf{k}}) \ \text{N m}$

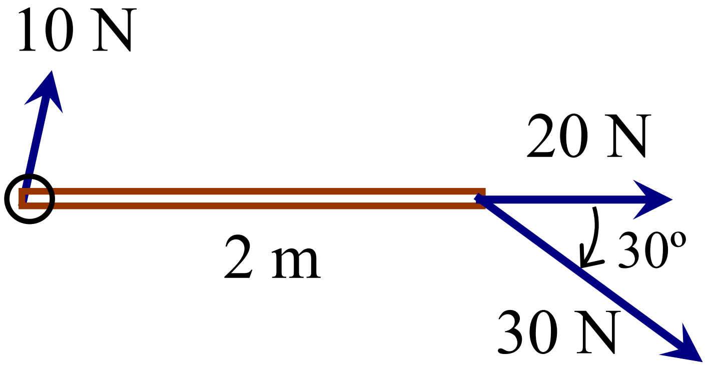
Examples

For each situation below, determine the resultant torque acting on the axis of rotation O. Use a coordinate system with the $\hat{\mathbf{k}}$- axis out of the page.

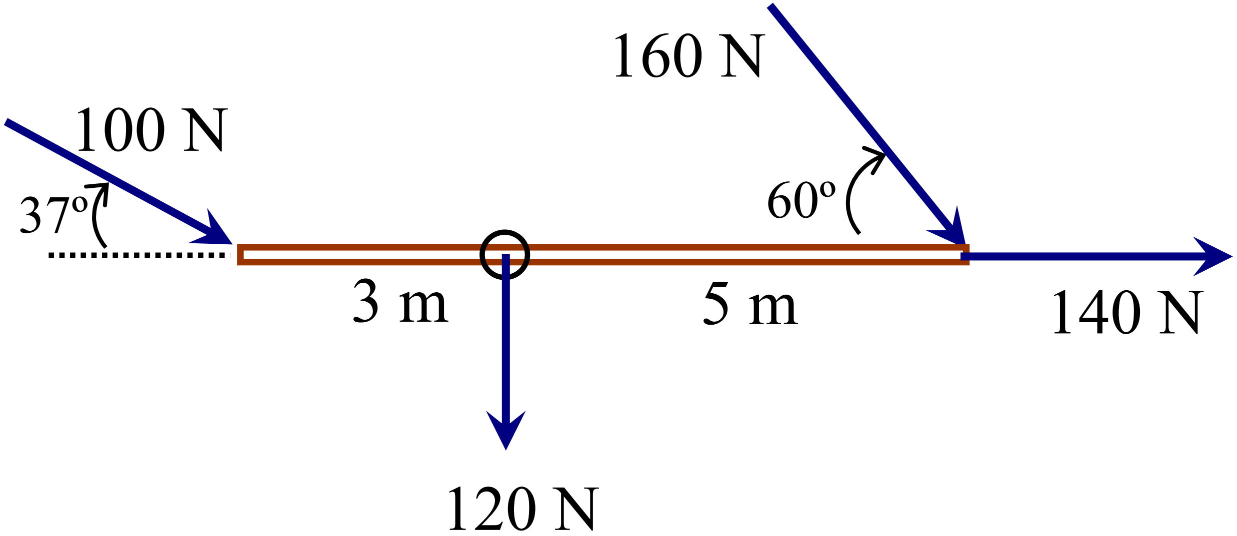




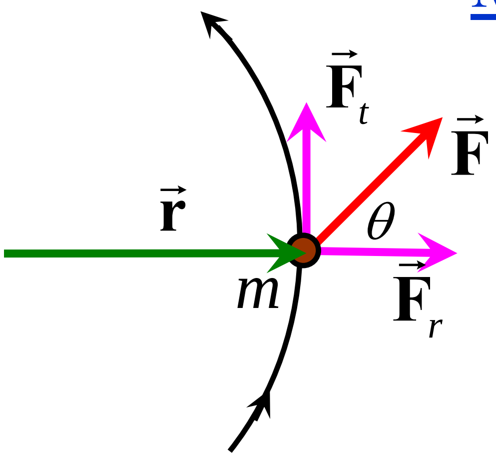
(c)



(d)



Moment of Inertia



$$\tau = r\ F \sin\theta$$
$$= r\ F_{\text{tangential}}$$

The tangential component of the force gives rise to a tangential acceleration, a_t , and therefore angular acceleration α.

$$\therefore F_t = ma_t = mr\alpha$$
$$\therefore \tau = rF_t = mr^2\alpha = I\alpha$$

where $I = mr^2$

I : **moment of inertia** (rotational inertia) of mass *m*, a distance r from the fixed axis of rotation.

I can be thought of as the 'resistance' of a body to being rotated.

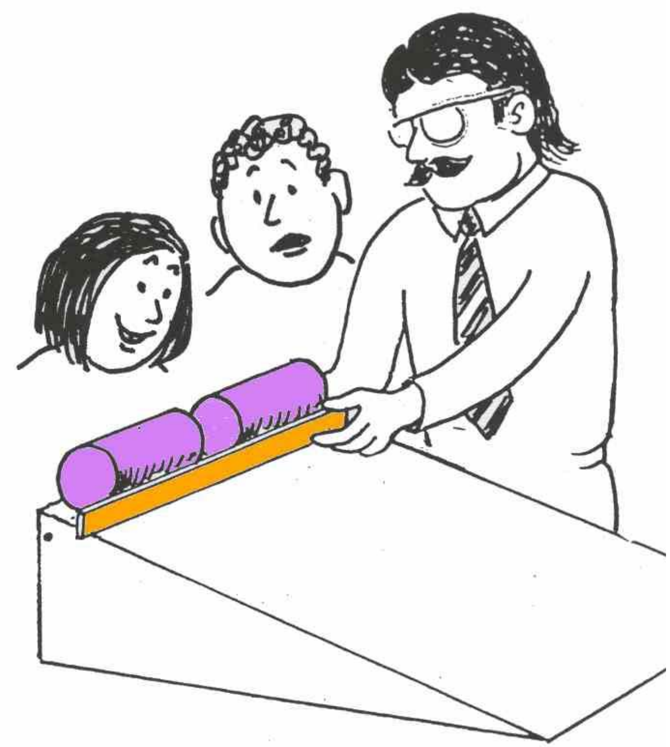
Unit of I : kg m^2

Figuring Physics

Roll a pair of identical cans of carbonated cooldrink down an incline. You won't be surprised to find they roll at the same rate. Now shake one of them so bubbles form inside, then repeat the experiment.

Now...

(A) the shaken can wins the race
(B) the shaken can loses the race
(C) both cans still roll together.



Moment of inertia cont.

We treat a rigid body (e.g. a plate, wheel) as consisting of many particles located at various distances from the axis of rotation.

Then the sum of the total torques

$$\sum \vec{\boldsymbol{\tau}} = \left(\sum m_i r_i^2 \right) \vec{\boldsymbol{\alpha}}$$

where r_i is the perpendicular distance of particle *i* from the fixed axis of rotation.

For a system of particles: $I = \sum m_i r_i^2$

For a rigid body: $I = \int r^2 dm$

Moment of inertia cont.

So we can see that the moment of inertia I is a measure of the rotational inertia of a body, and plays the same role for rotational motion that the mass does for translational motion. We can see that I depends both on the mass of the body and how that mass is distributed.

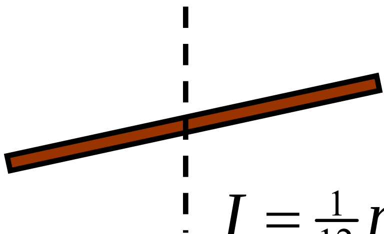
Moments of inertia can be calculated for any shape of body for rotation about any axis from the formula

$$I = \int r^2 dm$$

Some moments of inertia for rigid bodies

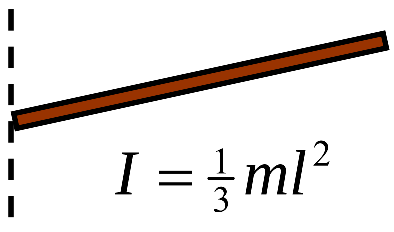
- - - axis of rotation

a thin rod of length *l*



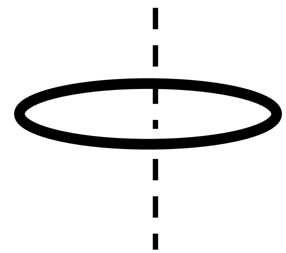
$$I = \tfrac{1}{12} m l^2$$

a thin rod of length *l*



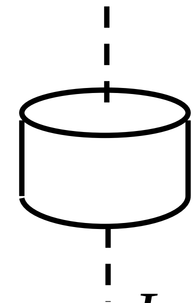
$$I = \tfrac{1}{3} m l^2$$

a thin hoop of radius *r*



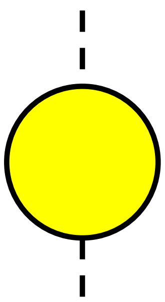
$$I = m r^2$$

a solid cylinder of radius *r*



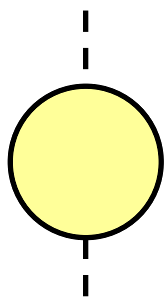
$$I = \tfrac{1}{2} m r^2$$

a solid sphere of radius *r*



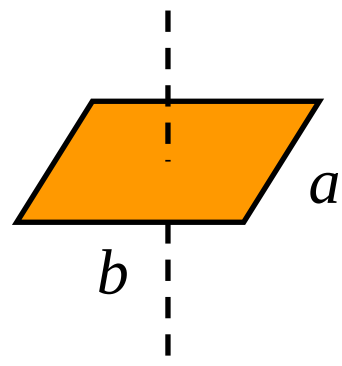
$$I = \tfrac{2}{5} m r^2$$

a hollow sphere of radius *r*



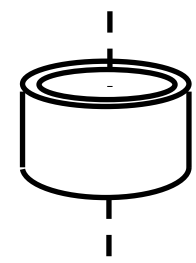
$$I = \tfrac{2}{3} m r^2$$

a thin plate



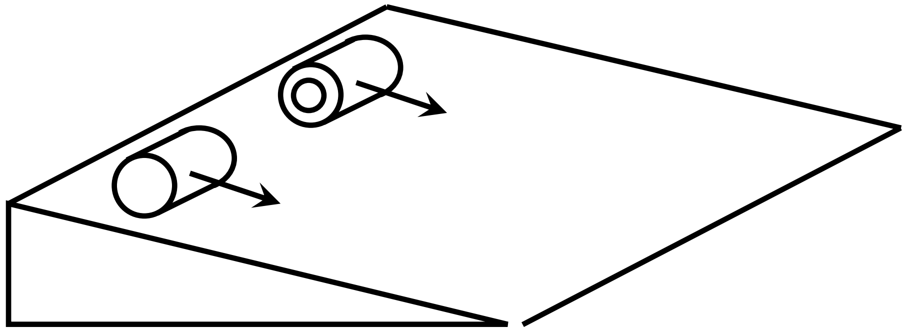
$$I = \tfrac{1}{12} m (a^2 + b^2)$$

a hollow cylinder of inner radius r_1



$$I = \tfrac{1}{2} m (r_1^2 + r_2^2)$$

Demonstration



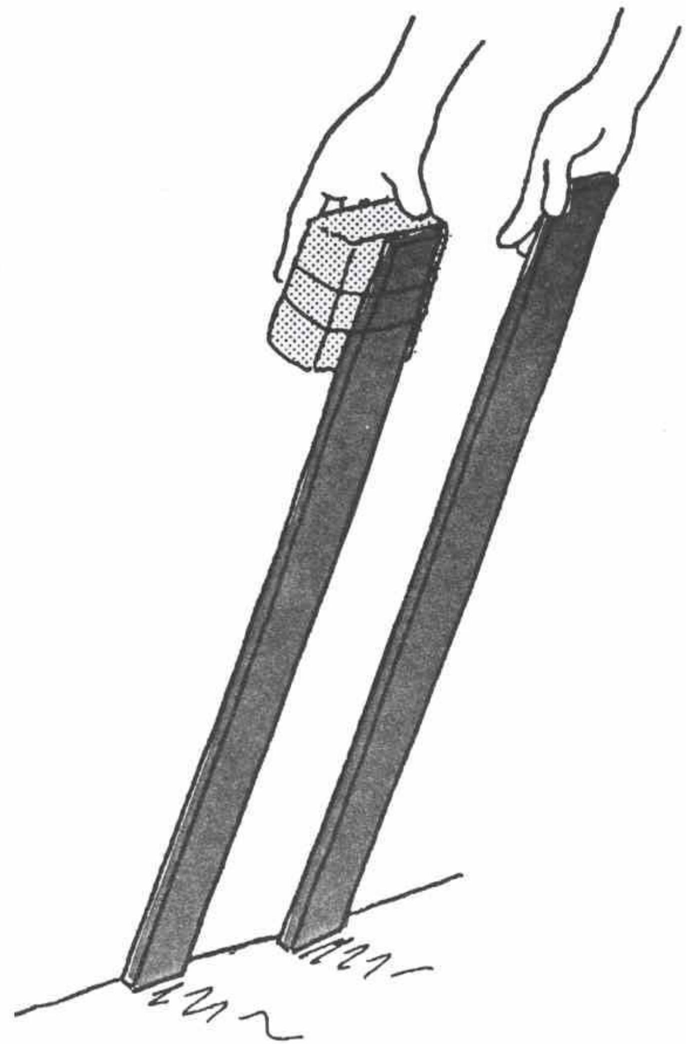
A solid cylinder and a hollow cylinder are raced down an incline.
Which reaches the bottom first, and why?

FIGURING PHYSICS

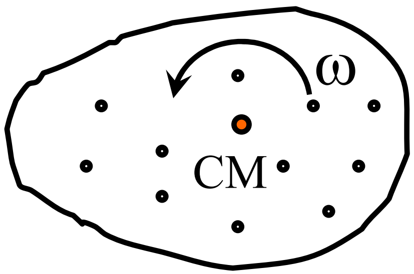
A pair of upright metre sticks, with lower ends against a wall, are allowed to fall to the floor. One is bare, and the other has a heavy weight attached to its upper end. The stick to hit the floor first is the

(A) bare stick
(B) weighted stick
(C) …both the same

Try it and see !



Rotational Kinetic Energy



For a rigid body rotating about fixed axis

$$K_{Rot} = \tfrac{1}{2} m_1 v_1{}^2 + \tfrac{1}{2} m_2 v_2{}^2 + \ldots$$

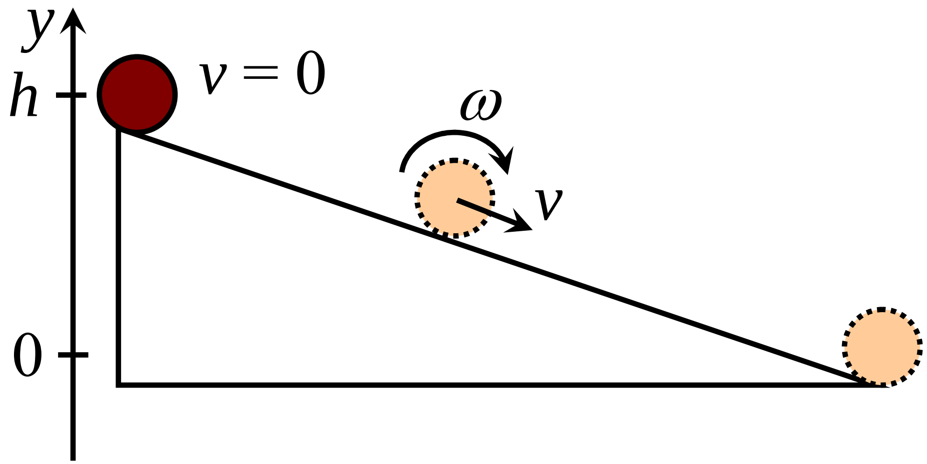
$$= \tfrac{1}{2} m_1 r_1 \omega^2 + \tfrac{1}{2} m_2 r_2 \omega^2 + \ldots$$

$$= \tfrac{1}{2} \left(\sum m r^2\right) \omega^2$$

$$\therefore K_{Rot} = \tfrac{1}{2} I \omega^2$$

A body that rotates while its CM undergoes translational motion will have K_{Total}:

$$K_{Total} = K_{Trans_{CM}} + K_{Rot} = \tfrac{1}{2} m v^2{}_{CM} + \tfrac{1}{2} I_{CM} \omega^2$$



Consider a solid sphere (mass m and radius r_0) rolling without slipping down an incline of height h.

Total energy at height y is $K_{trans} + K_{rot} + U_g = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + mgy$

Total energy at the top ($v = 0$, $w = 0$) $= m\,g\,h$

Total energy at the bottom $= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

I for a solid sphere for axis of rotation through centre $= \frac{2}{5}mr_0{}^2$

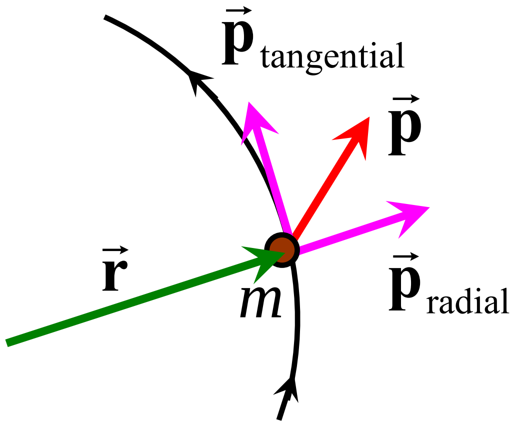
and since $v = v_{\text{tangential}} = r_0\omega$

Conservation of energy: $mgh = \frac{1}{2}mv^2 + \frac{1}{2}(\frac{2}{5}mr_0{}^2)\frac{v^2}{r_0{}^2}$

... giving ... $v = \sqrt{\frac{10}{7}gh}$

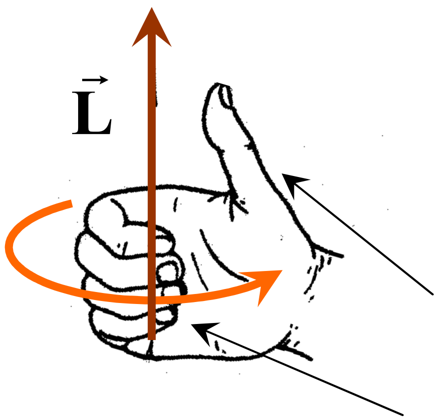
If the ball slipped (no friction): $v = \sqrt{2gh} > v_{roll} = \sqrt{\frac{10}{7}gh}$

Angular Momentum



$$\boxed{\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}}$$

$$\left|\vec{\mathbf{L}}\right| = r\ p\ \sin\theta$$
$$= r\ p_{\text{tangential}}$$
$$= r\ m\ v_{\text{tangential}}$$

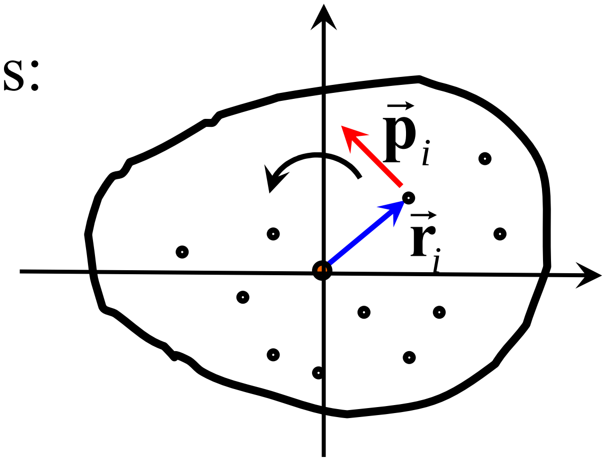


Direction of $\vec{\mathbf{L}}$ given by right-hand rule

thumb in direction of $\vec{\mathbf{L}}$

fingers of hand in direction of rotation

For a rigid body rotating about a fixed axis:



$$L = \sum l_i = \sum r_i p_i$$

$$= \sum r_i m_i v_i = \sum r_i m_i r_i \omega_i = \underbrace{\left(\sum m_i r_i^{\,2} \right)}_{I} \omega_i$$

$$\therefore \quad \vec{\mathbf{L}} = I \, \vec{\boldsymbol{\omega}}$$ unit: kg m^2 rad s^{-1}

Now, starting with Newton II:

$$\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt} \longrightarrow \vec{\mathbf{r}} \times \vec{\mathbf{F}} = \vec{\mathbf{r}} \times \frac{d\vec{\mathbf{p}}}{dt} \qquad \therefore\ \vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \frac{d\vec{\mathbf{p}}}{dt}$$

But $\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}} \longrightarrow \dfrac{d\vec{\mathbf{L}}}{dt} = \dfrac{d}{dt}(\vec{\mathbf{r}} \times \vec{\mathbf{p}})$

$$\therefore\ \frac{d\vec{\mathbf{L}}}{dt} = \left(\vec{\mathbf{r}} \times \frac{d\vec{\mathbf{p}}}{dt}\right) + \left(\frac{d\vec{\mathbf{r}}}{dt} \times \vec{\mathbf{p}}\right) = \underbrace{\left(\vec{\mathbf{r}} \times \frac{d\vec{\mathbf{p}}}{dt}\right)}_{\vec{\boldsymbol{\tau}}} + \underbrace{\vec{\mathbf{v}} \times m\vec{\mathbf{v}}}_{0}$$

or $\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \dfrac{d\vec{\mathbf{p}}}{dt} = \dfrac{d\vec{\mathbf{L}}}{dt}$

or $\vec{\boldsymbol{\tau}} = \dfrac{d}{dt}(I\vec{\boldsymbol{\omega}}) = I\,\vec{\boldsymbol{\alpha}}$

(Newton II for rotational motion)

Conservation of angular momentum

$$\vec{\boldsymbol{\tau}} = \frac{d}{dt}(I\vec{\boldsymbol{\omega}}) = I\,\vec{\boldsymbol{\alpha}}$$

If the net torque acting on a body is zero, i.e. $\sum \vec{\boldsymbol{\tau}} = 0$

then $\dfrac{d\vec{\mathbf{L}}}{dt} = 0$ or $\vec{\mathbf{L}} = I\vec{\boldsymbol{\omega}}$ = constant.

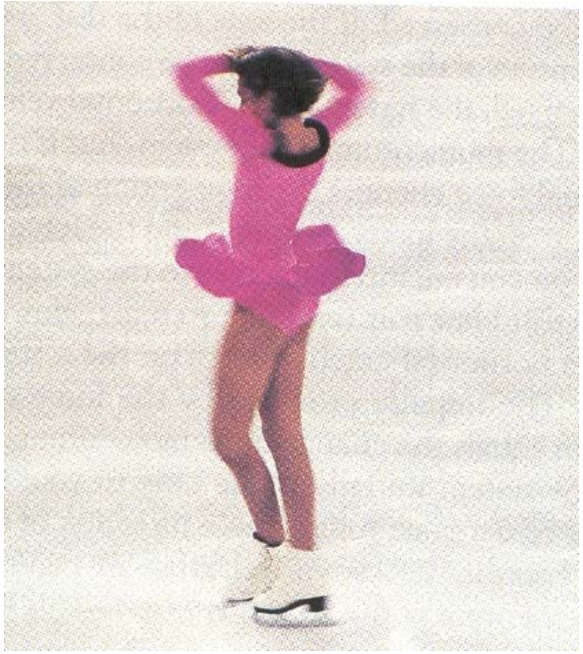
i.e. $\vec{\mathbf{L}}_{\text{initial}} = \vec{\mathbf{L}}_{\text{final}}$

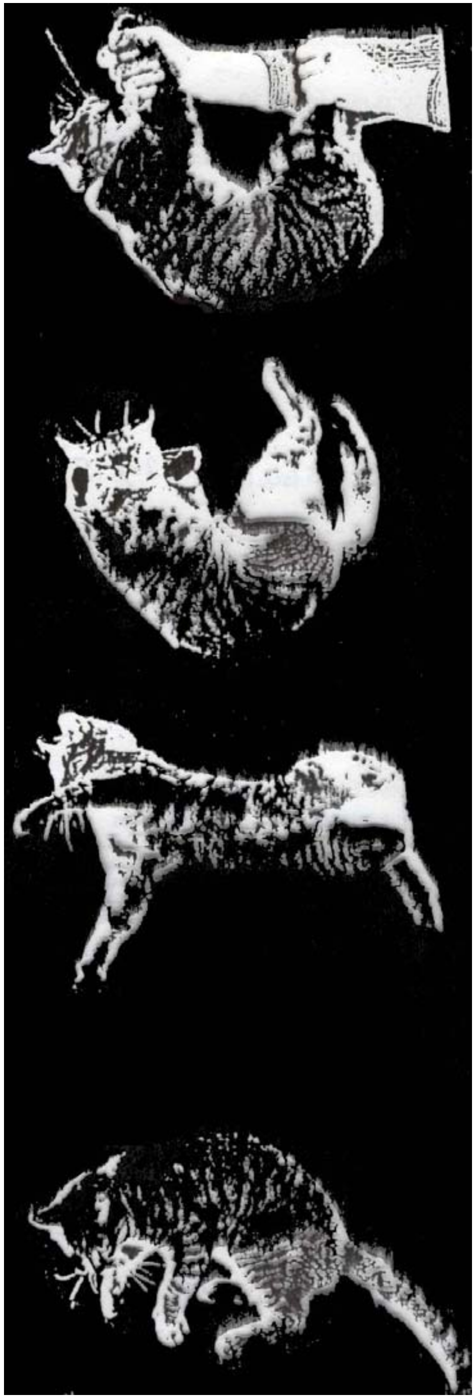
or $I_i\,\vec{\boldsymbol{\omega}}_i = I_f\,\vec{\boldsymbol{\omega}}_f$

Law of conservation of angular momentum

The total angular momentum of a rotating body remains constant if the net torque acting on it is zero.

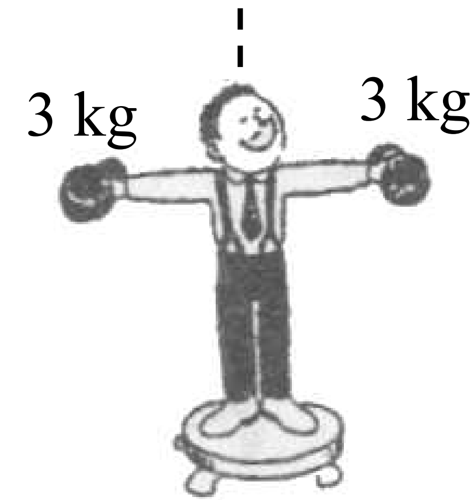
Conservation of angular momentum



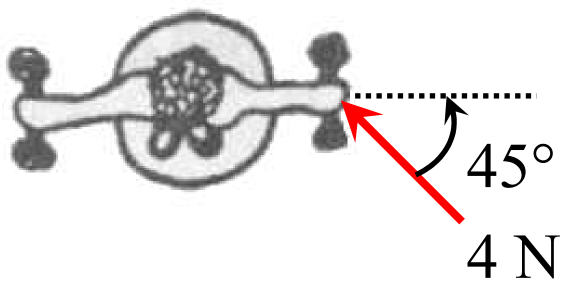


Example 1

A man stands on a rotating stool which is free to rotate without friction. He holds 3 kg in each hand at a distance of 1 m from the centre of this body.



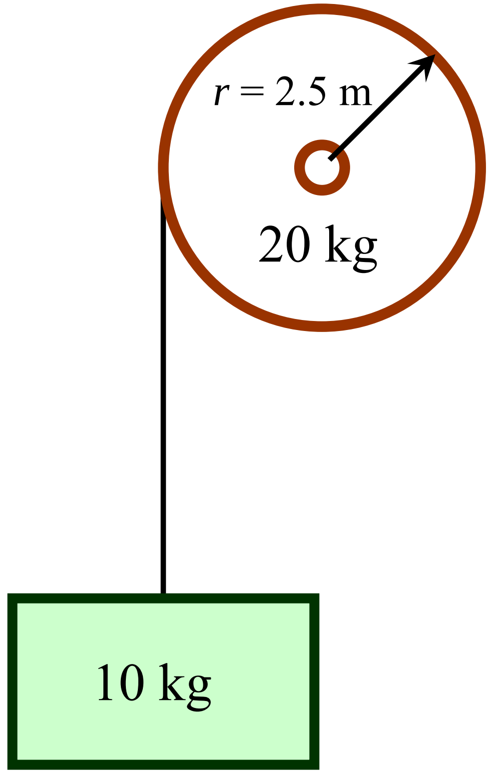
(a) Suppose that you give one of the masses a push if 4 N at an angle of 45° as shown and the man rotates. If you apply this force for only 0.5 s, what is the final angular velocity of the man and the weights? Use $I_{man} = 6$ kg m^2.



(b) If the man brings the masses straight towards his chest until they are a distance of 0.2 m from the centre of his body, what will now be his angular velocity?

Example 2

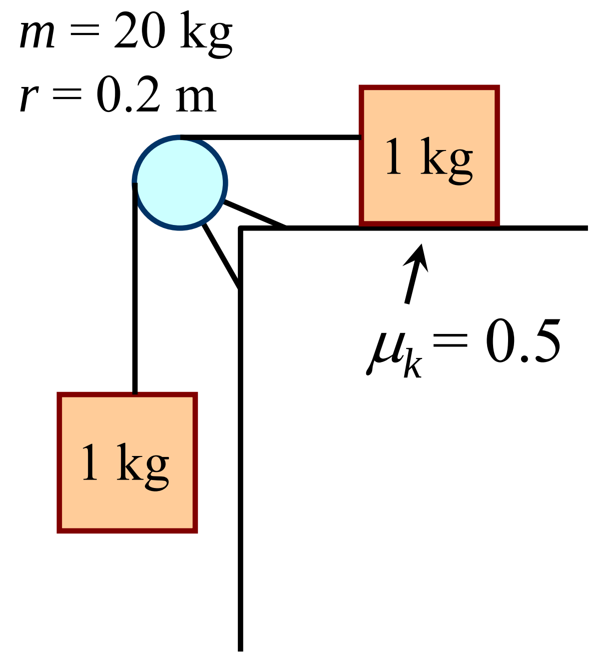
A 10 kg block is attached to a light rope which is wound around a 20 kg cylindrical pulley of radius $r = 2.5$ m.
How long will it take for the speed of the 10 kg block to increase from zero to 4.0 m s^{-1} after the system is released?
Use $I_{\text{cylinder}} = 6$ kg m^2.



Note that the pulley is <u>not</u> massless, hence we need to include the moment of inertia of the pulley.

Example 3

Two 1 kg blocks are connected to each other by a light string. The string loops over a pulley of mass 20 kg and radius 0.2 kg. If the coefficient of kinetic friction between the block and the surface is 0.5, what is the acceleration of the blocks?



Statics

... forces in equilibrium

The system is in equilibrium when the vector sum of all external forces acting on the system is zero.

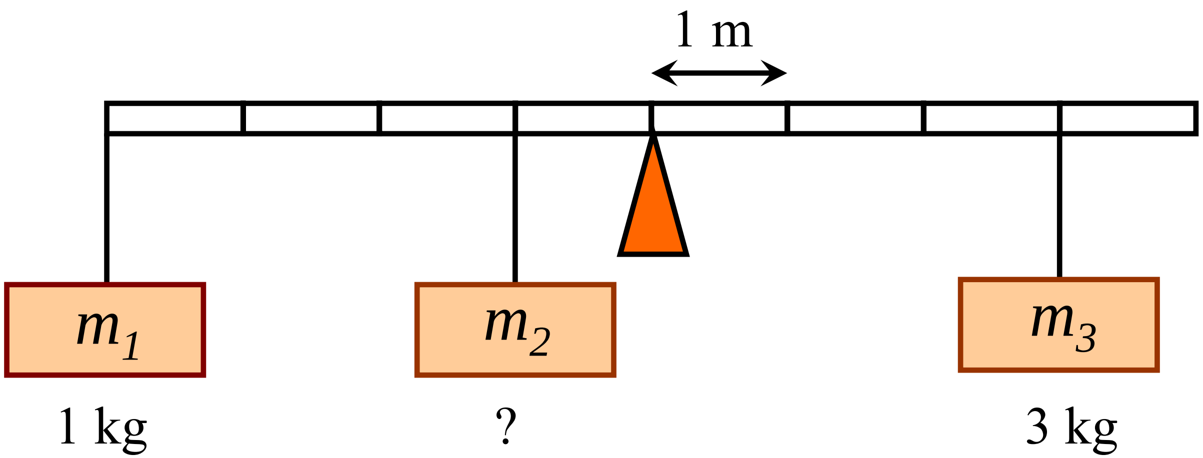
1st condition: $\sum \vec{\mathbf{F}} = 0$

i.e. $\sum \vec{\mathbf{F}}_{\hat{\mathbf{i}}} = 0\ ;\ \sum \vec{\mathbf{F}}_{\hat{\mathbf{j}}} = 0\ ;\ \sum \vec{\mathbf{F}}_{\hat{\mathbf{k}}} = 0$

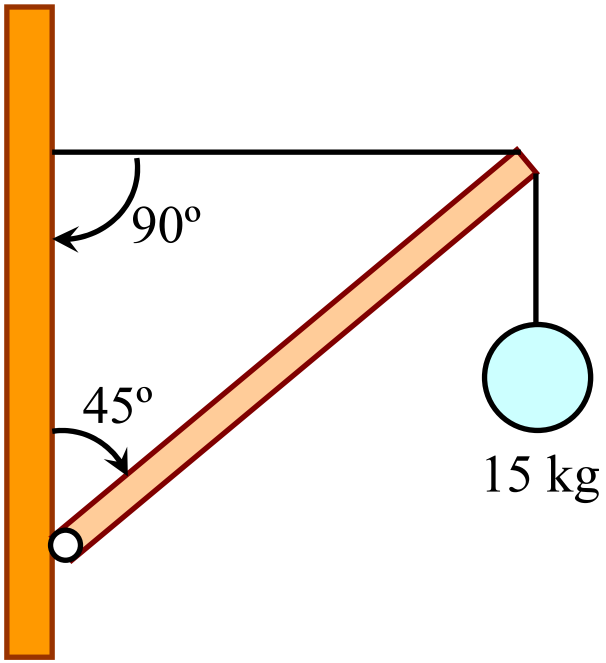
2nd condition: $\sum \vec{\boldsymbol{\tau}} = 0$ around <u>any</u> axis

Statics example 1

A beam is balanced on a fulcrum (triangle) with three masses hung from the positions shown.
What is the mass of m_2?
Assume that the beam is massless.

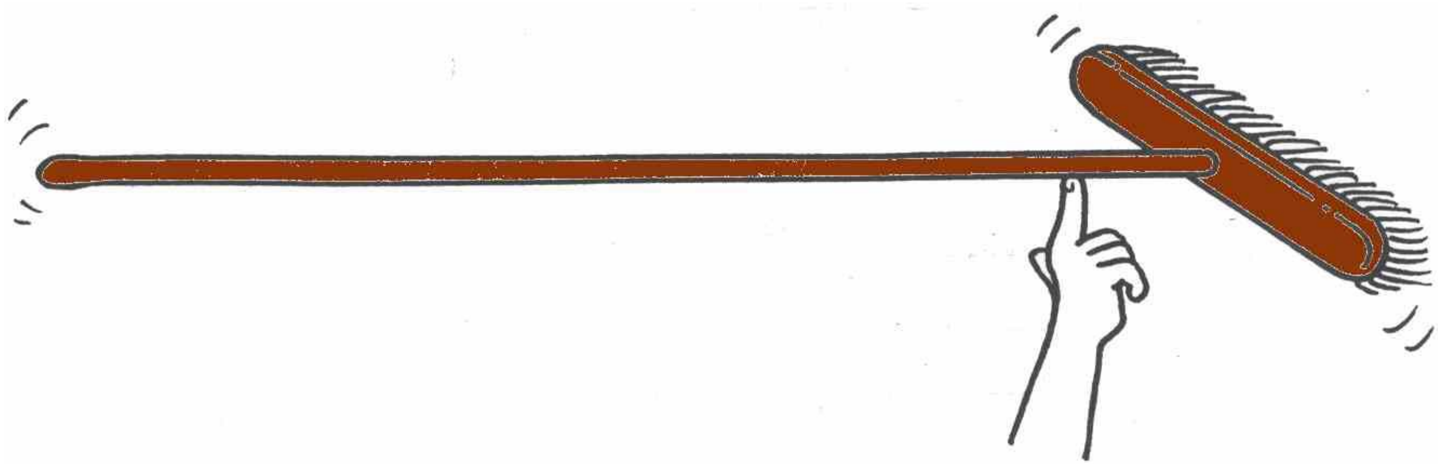


Statics example 2



A beam of mass 5 kg and length 10 m is held at an angle of 45° by a cable as shown. If a 15 kg mass hangs from the end of the beam, determine the force of the wall on the beam and the tension in the cable.

FIGURING PHYSICS



The broom balances at its centre of mass. If you cut the broom into two parts through the centre of mass and then weigh each part on a sale, which part will weigh more?

(A) the longer piece
(B) the shorter piece
(C) both the same